ELEMENTARY STATICS

A TREATISE

ON

ELEMENTARY STATICS.

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PREFACE.

THE following treatise is intended to give a simple explanation of the part of Statics required in the Previous Examination and the Second Examination for Ordinary Degrees.

The propositions requiring a knowledge of Trigonometry are marked with *Roman* numerals.

The Examples have been selected from Papers set in University Examinations.

A special acknowledgment is due from the Author to Mr Parkinson, of St John's College, for permission to make free use of his work on Mechanics.

ELEMENTARY STATICS.

ERRATUM.

page 3. line 24. *For* mines *read* vallies

. . . so that its length, breadth and thickness are less than any assignable linear magnitude.

A Body is made up of an indefinite number of particles.

A Rigid Body is a group of material particles held together in an invariable position with respect to each other.

2. Rest. When a body or particle constantly occupies the same position, it is said to be at rest.

Motion. When the position of a body or particle is being changed continuously, it is said to be in motion.

3. Force. Any cause which changes or tends to change the state of rest or motion of a body or particle is called force.

Line of Action. The line of action of a force is the line in which a particle would begin to move in consequence of the action of the force.

EQUILIBRIUM. If several forces acting on a particle or on a body are so related that no motion of the particle or the body takes place, the forces are said to be in equilibrium.

PRESSURE. If one body press against another, each body is subjected to a force acting at the point of contact: such force is called pressure.

TENSION. When a body is pulled by means of a string or pushed by means of a rod, the force exerted along the string or rod is called tension.

WEIGHT or GRAVITY is the name given to the force with which the earth attracts a body.

4. DENSITY. The closeness with which particles are packed in a body is called density.

VOLUME or MAGNITUDE is the amount of space occupied by a body.

In forming notions of Weight and Volume we are assisted by the faculties of muscular action, of touch and sight. In estimating the Weight and Volume of particular bodies we refer to certain definite standards of Weight and Volume, as a pound and a cubic inch.

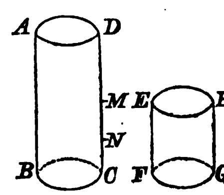
But of Density we can form no distinct idea, because we are ignorant of the nature of matter. We know by experience that in some bodies the component particles lie closer than in others, and we can estimate the closeness with which the particles are packed in a particular substance by comparing the weight of a unit of volume of that substance with the weight of a unit of volume of some standard substance. Still in this mode of proceeding we make an assumption which we cannot prove, viz., that the elementary particles of which the two substances are composed are alike in form and weight.

5. *On Mass and Weight.*

It is important to distinguish between Mass and Weight.

The Mass or Quantity of Matter in a body varies as the product of the Volume and Density of the body.

To explain this let us suppose that $ABCD$, $EFGH$ are two cylindrical vessels with equal bases. They stand side by side on a table, and M, the middle point of the vertical line CD, is on a level with EH, the top of the smaller vessel. Then the volume of $ABCD$ is twice the volume of $EFGH$.



Now pour snow lightly into each vessel till the surface reaches M and EH. Press the snow down in the larger vessel till its surface is at N the middle point of CM. Then fill the larger vessel with snow condensed in the same degree as that in CN.

Both vessels are now full of snow, and the mass of the snow in the larger vessel is *four times* as great as the mass of the snow in the smaller vessel, because it occupies *twice as large a volume and is twice as dense.*

The mass of a body will be the same at all parts of the Earth's surface, but the weight of a body differs in different latitudes. For the force of gravity, which affects the weight but not the mass of a body, is greater or less as the body is nearer to or further from the centre of the Earth: greater in mines, less on mountains: greater at the Poles, less at the Equator.

There are two balances very commonly employed for weighing letters.

One is the scale-balance, in which a letter is put in one scale and a standard weight in the other.

The second is the spring-balance, in which a letter is placed on the top of the machine, and its weight is estimated by the distance through which it depresses a spring.

Now weighed in the scale-balance a letter would have the same apparent weight at the Equator and at London, but with the spring-balance the apparent weight would be less at the Equator than at London. The Mass of the letter would be the same in both places.

But at *any given locality* the weight of a body is a practical measure of its mass.

6. *Method of estimating forces.*

The three elements specifying a force, all of which must be known in order to estimate the effect of the force, are

(1) The point of application of the force.

(2) The direction in which the force acts.

(3) The magnitude of the force.

If two forces be applied in *opposite* directions to a point which is free and at rest, and constitute an equilibrium, they are said to be equal forces.

If two equal forces be applied in the *same* direction to the same point, we shall have a *double* force; if in the same way we combine three equal forces, we shall have a *triple* force, and so on: so that, in general, to *measure* forces we must take some known force as unit, and then express in *numbers* the relation which the other forces bear to this unit.

It is usual to take as the unit of force that force which will sustain, when acting vertically upwards, a weight of one pound: a force which will sustain two pounds will then be represented by the figure 2, a force which will sustain three pounds by the figure 3, and, generally, a force which will sustain P pounds will be represented by P.

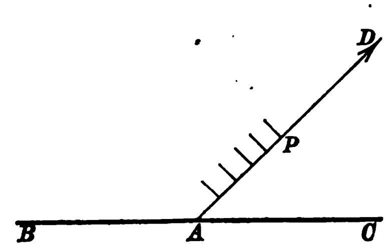
7. *Method of representing forces.*

Forces may be represented by straight lines: for

(1) A straight line can be drawn from any point, and thus it will represent a force with respect to the point of application.

(2) A straight line can be drawn in any direction, and thus it will represent the direction of a force.

(3) A straight line can be drawn of such a length as to contain as many units of length as the given force contains units of force, and thus it will represent the magnitude of a force.



Thus, suppose we are speaking of a force of 5 lbs. acting at the middle point of a horizontal rod and inclined at an angle of 45° to the horizon.

Let BC represent the rod, A the middle point of the rod.

Draw AD making an angle of 45° with AC.

Mark off a portion of the line AD, suppose AP, containing 5 units of length, that is, as many units of length as there are units of force in the given force.

Then we may say that AP represents the given force in every particular:

(1) In point of application, at A the middle point of the rod.

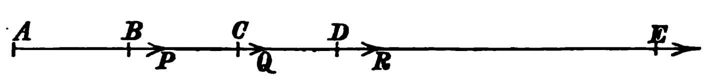
(2) In direction, as being inclined at an angle of 45° to the horizon.

(3) In magnitude, by the number of units in its length.

8. It is to be carefully observed that the *order* of the letters

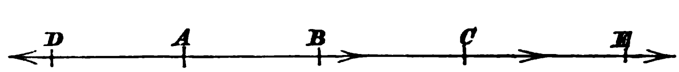
indicates the direction of the force. Thus AB expresses that the force represented by the line AB acts in the direction of the arrow from A to B. A force represented by BA would be of equal magnitude, but acting in the opposite direction, that is, from B to A.

Now suppose we have several forces, as P, Q, R acting simultaneously at the point A in the same direction: if separately they



would be represented by AB, AC, AD: they will when acting simultaneously be represented by a line AE, the length of which is equal to the sum of $AB+AC+AD$.

If one of the forces, as R, acts in a direction opposite to that of

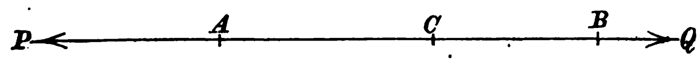


the other two, P and Q, we shall have to subtract the line AD from the sum of the others, AB, AC, and the three will be represented by a line AE equal in length to $AB+AC-AD$.

This is still the algebraic *sum* of the lines AB, AC, AD, if lines in one direction from A be considered *positive*, and lines in the opposite direction *negative*.

9. *On the Transmissibility of force.*

It is plain that two equal and opposite forces, P, Q, applied at the



extremities of a straight rigid rod AB, and acting in direction of the rod, will be in equilibrium.

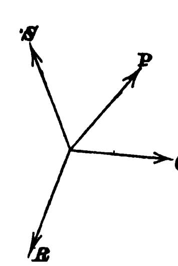
This result will be true whatever be the length of the rod: and hence we infer that P will balance Q at whatever point of the rod Q be applied; in other words, the effect of Q is the same at whatever point of the rod B, C,...... it may be applied, the direction remaining the same.

These considerations lead us to the following principle, called the *principle of the transmission of force.*

The effect of a force on a particle to which it is applied will be the same, if we suppose the force to be applied at any point in the line of action, provided the point be rigidly connected with the original particle.

10. *On Component and Resultant Forces.*

When a system of forces acting on a particle at rest is not in equilibrium, the particle will begin to move in some definite direction, but a single force might be found of proper intensity, which when applied to the particle and acting *in the same direction* would cause the particle to move in exactly the same manner: such a force is called the *Resultant* of the system of forces, and the constituent forces of the system, with reference to this resultant, are called *Components.*

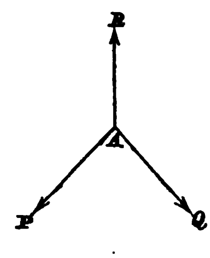


11. When a system of forces as P, Q, R, S is in equilibrium, one of them, as P, may be regarded as counterbalancing the combined action of all the rest, Q, R, S.

It appears then that the remaining forces Q, R, S, produce the same effect on the particle as would result from a single force equal and opposite to P. We infer then, that when a system of forces acting on a body is in equilibrium, *any one* of the forces is equal and opposite to the resultant of all the rest.

12. If P and Q be two forces whose lines of action meet in the point A, it is easy to see

(1) That they have some resultant, equal and opposite to the force R, which when acting with P and Q keeps A at rest.

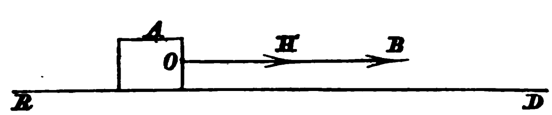


(2) That this resultant must lie within the angle PAQ which is less than 180°.

(3) That when P and Q are equal the direction of the resultant bisects the angle between the direction of the two components P, Q.

13. *Illustrations of Component and Resultant Forces.*

Since a clear conception of the meaning of the terms *Component* and *Resultant* is necessary for a right understanding of Statical principles, we shall give in this and the following article two rough illustrations which may serve to explain the definition given in Art. 10.



A is a block of stone to be drawn along a level road RD.

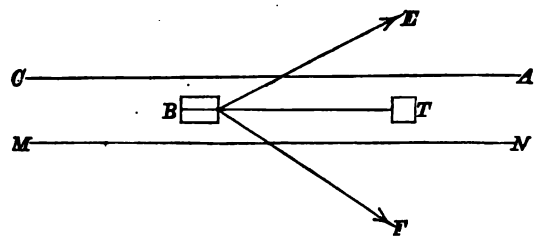
Suppose two horses of equal power to be attached to the stone at O in such a manner that each exerts a force along the line OHB, one at H and the other at B.

Now suppose the horses to be removed, and a traction-engine of two-horse power to be applied to the block at O so as to move it in the direction OHB with the same drawing-power as that of the two horses.

The traction-engine will represent the Resultant.

The horses will represent the Component Forces.

14.



CA, *MN* are the straight and parallel edges of the banks of a river.

B is a barge in the middle of the stream.

Suppose two horses of equal power to be pulling at ropes attached to the same point of the barge, the ropes being inclined at equal angles to the line *BT* which passes along the middle of the stream, parallel to the banks.

Then the barge will move along the line *BT*.

Now suppose the horses to be removed and a steam-tug to be attached to the barge so as to move it in the direction *BT* just in the same manner as it moved when the horses were pulling it.

The steam-tug will represent the Resultant.

The horses will represent the Component Forces.

15. From the two illustrations of Component and Resultant Forces which have been given we may derive examples of forces in Equilibrium.

For, first, suppose that while the horses are pulling at the stone the traction-engine is applied *to the opposite side* of the stone, so as to pull the stone with the same drawing-power as that of the horses.

Then the force exerted by the engine will counteract the forces exerted by the horses, and the three forces will be in equilibrium.

Then also it is plain that when three forces are in equilibrium, one of them is equal and opposite to the resultant of the other two.

Precisely the same results will follow if in the second illustration we suppose the steam-tug to be applied *to the opposite end* of the barge.

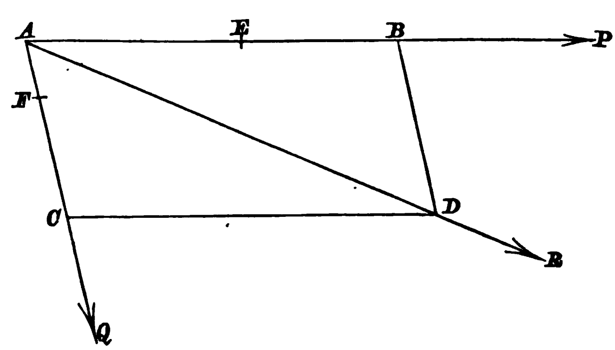
N.B. When two or more forces are in equilibrium their Resultant is *Zero*.

16. THE PARALLELOGRAM OF FORCES.

We proceed to establish an important theorem which enables us to determine the resultant of any two forces acting at a point. The theorem is called the Parallelogram of Forces, and may be thus enunciated.

If two forces acting at a point be represented in magnitude and direction by two straight lines drawn from that point, and if a parallelogram be constructed having these two lines for adjacent sides, then that diagonal of the parallelogram which passes through the point of application of the forces will represent their resultant in magnitude and direction.

That is, if the two forces P, Q be represented by AB, AC and the parallelogram $ABDC$ be completed, their resultant R will be represented by the diagonal AD.



The same is true if P, Q act at points E, F, provided their directions meet in some point A.

We shall divide the proposition into two parts

I. To prove that the resultant acts *in direction* of the diagonal.

II. To prove that the diagonal represents *the magnitude* of the resultant.

Part I. is subdivided into two cases.

(1) When the forces are commensurable.

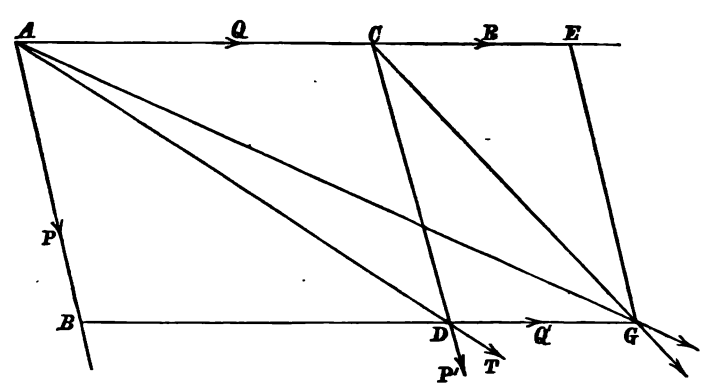
(2) When the forces are incommensurable.

17. **Part I. Case (1).** To prove that the resultant acts in the direction of the diagonal, the forces being *commensurable*.

When the forces are equal ($AB = AC$), their resultant bisects the angle between the direction of the forces, and therefore acts along the diagonal AD: that is, this first part of the proposition is true *for two equal forces*.

Let us assume (a) for the present that it is also true for two sets of forces P and Q, P and R, equal or unequal; we can then prove that it is true for the forces P and $Q+R$.

Let P act at A in direction AB, Q and R act at A in direction ACE, and let AB, AC represent P, Q in magnitude, and since R may be supposed to act at any point in the line ACE which is rigidly connected with A, let R act at C, and let CE represent R in magnitude.



Complete the parallelograms BC, DE.

Then since by the hypothesis (a) the resultant T of P, Q acts along AD, let them be replaced by their resultant, and let this resultant be applied at D, which may be done without altering its effect.

Now this resultant T, acting at D, may be decomposed into two forces P', Q' (equal respectively to P, Q) acting at D in directions CD, DG, which are parallel to AB, AC.

Let T be replaced by P', Q', and let the point of application of P' be removed to C, and that of Q' to G.

Again, P' and R acting at C have a resultant acting in direction CG, by hypothesis (α): let them be replaced by this resultant, and let its point of application be transferred to G.

We have thus shewn (on the hypothesis α) that the forces P, Q, R which are applied at A, may be supposed to be applied at G without altering their combined effect,—that is, AG must be the direction of the resultant of P and $Q+R$ in any case in which the hypothesis (α) holds true.

Now this hypothesis *is* true when P and Q are each equal to the same force f, and it *is* true when P and R are each equal to the same force f; therefore the conclusion is true for P and $Q+R$ when P, Q, R are each equal to f, that is, it is true for f and $2f$.

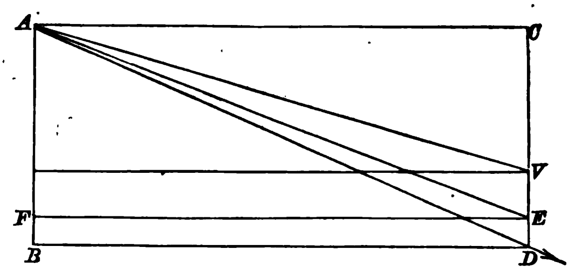
Again, since it is true for f and $2f$ and also for f and f, it is true for f and $2f+f$, that is for f and $3f$: and so by induction it is true for f and mf.

Again, putting $P=mf$, $Q=f$, $R=f$, our conclusion is true for two forces mf and $2f$, and again for mf and $3f$, and so by induction it is true for mf and nf, m and n being any integers whatever.

Now any two commensurable forces may, by assigning a proper value to f, be expressed by mf, nf.

Hence Case (1) is proved.

18. Part I. Case (2). To prove that the resultant acts in the direction of the diagonal, if the forces are *incommensurable.*



Let AB, AC represent two incommensurable forces.

Complete the parallelogram $ABDC$, and if AD be not the direction of the resultant, let it be some other line, as AV.

Let AC be divided into an integral number of equal parts each less than DV, which is always possible, and mark off from CD portions equal to these, the last division E clearly falling between D and V.

Complete the parallelogram CF by drawing EF parallel to AC.

Then the resultant of AC, AF will be in direction AE, and we may suppose this resultant to be substituted for them.

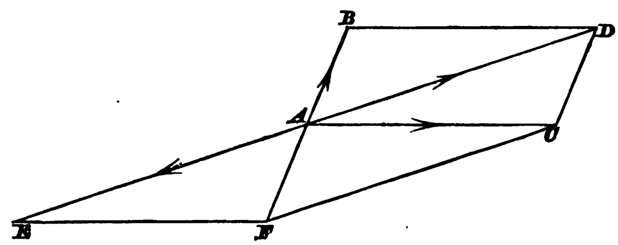
The resultant then of AC and AB is equivalent to the resultant of some force in the direction AE together with FB which acts along AB, and this resultant must lie *within* the angle BAE.

But by hypothesis it acts in the direction AV, *without* the same angle, which is absurd.

In like manner it may be shewn that no direction but AD can be that of the resultant of the forces AB, AC.

Thus the theorem has been proved so far as *the direction of the resultant* is concerned.

19. PART II. *We have now to prove that the diagonal represents the resultant in magnitude.*



Let AB, AC represent the component forces in magnitude and direction.

Complete the parallelogram $ABDC$: join AD.

In DA produced take AE of such a length as to represent the magnitude of the resultant of AB, AC.

Complete the parallelogram $AEFC$: join AF.

Now AB, AC, AE represent three forces which balance each other.

Therefore AB represents a force equal *and opposite* to the resultant of AC, AE.

But the resultant of AC, AE *lies in the direction of* AF.

Therefore AB is in the same straight line with AF.

Therefore $AFCD$ is a parallelogram;

and $$\therefore AD = FC;$$

but $$FC = AE;$$

$$\therefore AD = AE;$$

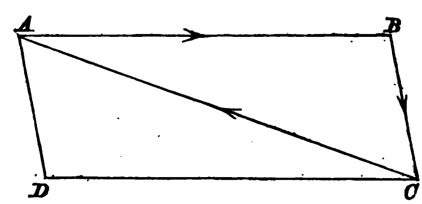
$\therefore$ AD represents in magnitude the resultant of AB, AC.

20. THE TRIANGLE OF FORCES.

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

Let AB, BC, CA, the sides of the triangle ABC taken in order, represent in magnitude and direction three forces applied at the point A.

Complete the parallelogram $ABCD$.



Then, since AD is equal and parallel to BC, the force represented in magnitude and direction by BC will be represented, acting at A, by AD.

Now the resultant of AB, AD is a force represented by AC.

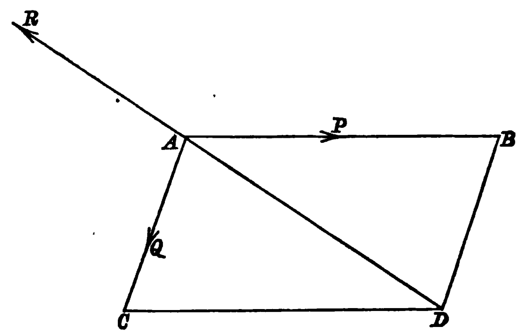
Therefore the three forces represented by AB, BC, CA, all applied at A, are equivalent to AC, CA, which will clearly balance each other.

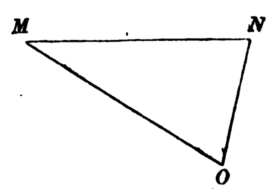
Therefore the three forces represented by AB, BC, CA, applied at any point A, will be in equilibrium.

21. *Converse of the Triangle of Forces.*

If three forces acting at a point balance each other, and any triangle be constructed having its sides parallel to the directions of the forces, the sides of the triangle shall be proportional to the forces.

Let P, Q, R be three forces which, acting at the point A, balance each other.





Let AB, AC represent P and Q.

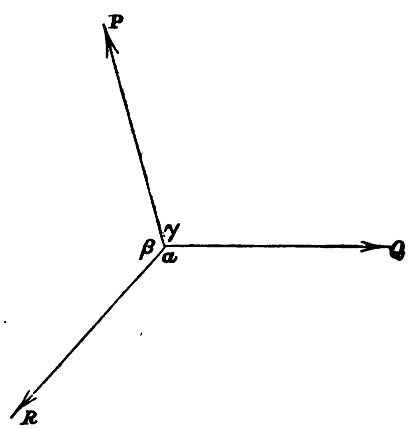
Then DA, the diagonal of the parallelogram $ACDB$, will represent R.

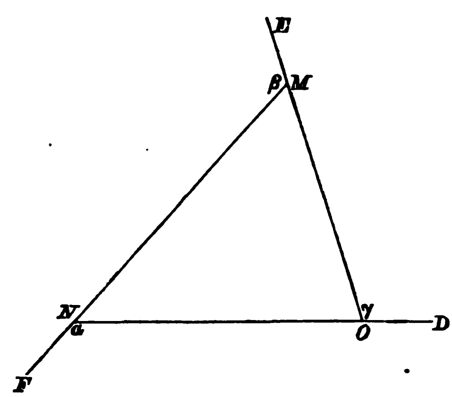
Now construct a triangle MNO whose sides are parallel to the sides of the triangle ABD.

Then ABD, MNO are similar triangles.

$$\begin{array}{lrcl}
\text{Hence} & MN : NO & :: & AB : BD \\
 & & :: & P : Q \\
\text{and} & NO : OM & :: & BD : DA \\
 & & :: & Q : R \\
\text{and} & OM : MN & :: & DA : AB \\
 & & :: & R : P.
\end{array}$$

xxii. *If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle contained between the directions of the other two.*





Let P, Q, R be the three forces.

α, β, γ the angles between the lines of direction of the forces.

Construct a triangle MNO whose sides MO, ON, NM are parallel, and therefore proportional, to P, Q, R.

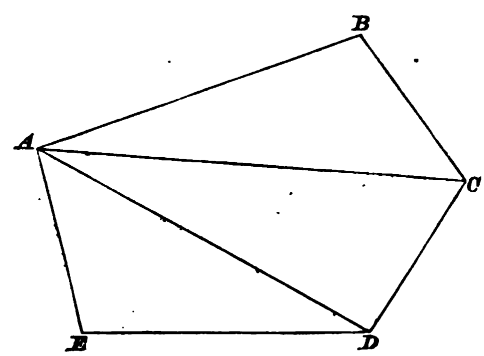
Produce the sides to D, E, F.

Then the exterior angles ONF, NME, MOD are equal to α, β, γ respectively.

$$
\begin{aligned}
\text{Now} \quad P : Q : R &:: MO : ON : MN \\
&:: \sin MNO : \sin NMO : \sin MON \\
&:: \sin ONF : \sin NME : \sin MOD \\
&:: \sin \alpha : \sin \beta : \sin \gamma.
\end{aligned}
$$

23. THE POLYGON OF FORCES.

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.



Let the forces acting at A be represented in magnitude and direction by the sides of the polygon $ABCDE$, taken in order, thus AB, BC, CD, DE, EA.

Join AC, AD.

Now AC is the resultant of AB, BC;

$\therefore$ AB, BC, CD will be represented by AC, CD.

Again, AD is the resultant of AC, CD;

$\therefore$ AB, BC, CD, DE will be represented by AD, DE.

Again, AE is the resultant of AD, DE;

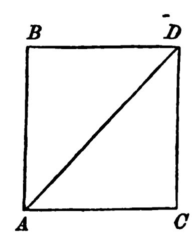
$\therefore$ AB, BC, CD, DE, EA will be represented by AE, EA.

Now AE, EA balance;

$\therefore$ AB, BC, CD, DE, EA will be in equilibrium.

The converse of this Theorem is not *necessarily* true, because the theorem that the sides of equiangular *triangles* are proportional cannot be extended to equiangular *polygons*. Thus a square and an oblong are equiangular, but the sides about the equal angles are not proportional.

24. *Two forces act upon the same point in directions at right angles to each other, to find the magnitude and direction of their resultant.*



Let AB, AC represent two forces acting at right angles to each other at the point A.

Complete the rectangular parallelogram $ABDC$.

Then the diagonal AD will represent the resultant of AB, AC.

Now, since the angle DCA is a right angle,

$$AD^2 = AC^2 + CD^2;$$

$$\therefore AD^2 = AC^2 + AB^2;$$

$$\therefore AD = \sqrt{AC^2 + AB^2};$$

and thus we obtain the *magnitude* of the resultant.

The *direction* of the resultant is known if we know the angle DAC.

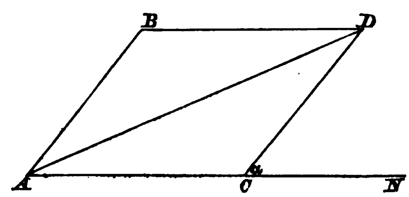
For certain simple relations between the sides of the triangle ADC we can determine the angle DAC by *geometry*. Thus if AB and AC are equal forces, AC and CD are equal, and DAC is half a right angle.

By the aid of *trigonometry* we can always determine the value of the angle DAC from the known values of the lines AB, AC.

For $$\tan DAC = \frac{DC}{CA} = \frac{AB}{AC}.$$

XXV. *Two forces act upon the same point and the angle between their lines of direction is given, to find expressions for the magnitude and direction of their resultant.*

Let AB, AC represent two forces acting upon the point A, and let α be the angle between their lines of direction.



Complete the parallelogram $ABDC$, and produce AC to N.

Then the angle $DCN = \alpha$. Join AD.

Then AD will represent the resultant of AB, AC.

Now we know by Trigonometry that

$$AD^2 = AC^2 + CD^2 - 2AC \, . \, CD \, . \cos ACD;$$

also, $$\cos ACD = -\cos DCN$$

$$= -\cos \alpha;$$

$$\therefore \; AD^2 = AC^2 + AB^2 + 2AC \, . \, AB \, . \cos \alpha;$$

and thus we obtain an expression for the *magnitude* of the resultant.

The *direction* of the resultant is known if we know the angle DAC.

Now $$\frac{\sin DAC}{\sin DCA} = \frac{DC}{AD},$$

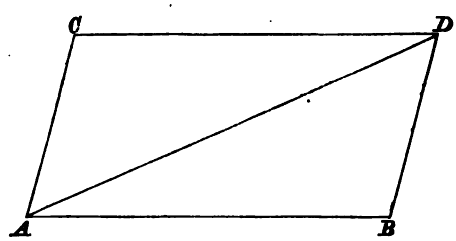
and $$\sin DCA = \sin DCN = \sin \alpha;$$

$$\therefore \; \frac{\sin DAC}{\sin \alpha} = \frac{DC}{AD},$$

or $$\sin DAC = \frac{DC}{AD} \, . \sin \alpha,$$

from which the value of the angle DAC may be determined.

26. We have seen, by the Parallelogram of Forces, that two forces AB, AC acting at a point A are equivalent to a single force AD, the diagonal of the parallelogram $ABDC$, acting at the same point.



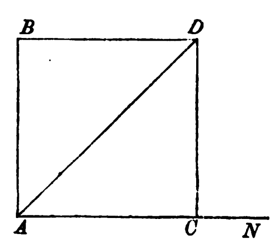
Thus we can *compound two forces into one.*

Conversely, if a line AD represent a force, and *any* parallelogram be constructed having AD for a diagonal, the single force AD may be replaced by two forces represented by AB, AC.

Thus we can *resolve one force into two.*

Also, since the number of parallelograms which can be constructed with AD as diagonal is unlimited, it follows that a single force can be resolved into two other forces equivalent to it in an unlimited number of ways.

27. *To resolve a given force into two component forces at right angles to each other.*



If we know the direction of one of the component forces, that is if we know the angle which it makes with the line of direction of the given force, we can determine its magnitude, and also the magnitude and direction of the other component.

For let AD represent the given force.

Draw AN in the known direction of one of the component forces.

Draw DC at right angles to AN,

AB at right angles to AC,

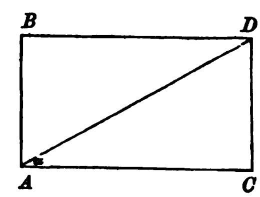
DB parallel to AC.

Then AB and AC will represent the component forces in magnitude and direction.

xxviii. With the aid of Trigonometry we can establish formulæ by which we can express in general terms the method of resolving a given force into two component forces with respect to which we merely know that they are to be at right angles to each other.

Let AD be the given force.

AB, AC the adjacent sides of any rectangular parallelogram of which AD is the diagonal.



Then AB, AC represent the Effective Components of the force AD estimated in the directions AB, AC respectively.

Now if we represent the angle CAD by α,

$$AC = AD \cdot \cos \alpha$$

$$AB = CD = AD \cdot \sin \alpha.$$

That is, we have obtained the following information:

A Force in any direction may be resolved in and perpendicular to any other direction. The first component is found by multiplying the force by the cosine of the angle between the two directions—the second by multiplying the force by the sine of the same angle.

It may here be observed that the following values of the Trigonometrical Ratios of certain angles should be committed to memory, because they are of frequent occurrence in examples in elementary statics.

$$\sin 90^\circ = 1, \qquad \cos 90^\circ = 0,$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2},$$

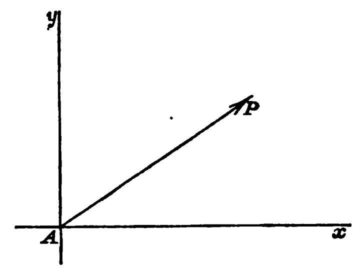
$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2},$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}},$$

$$\sin 0^\circ = 0, \qquad \cos 0^\circ = 1.$$

xxix. *We may now proceed to find the resultant of any number of forces acting in one plane at a point.*

Through the point A draw two lines Ax, Ay at right angles to each other in the plane in which the forces act.



Let α be the angle which one of the forces, P, makes with Ax.

Then P is equivalent to $P.\cos\alpha$ acting in the direction of Ax,

together with $P.\sin\alpha$ acting in the direction of Ay.

Similarly if P' be another of the forces, making an angle α' with Ax,

P' is equivalent to $P'.\cos\alpha'$ acting in the direction Ax

together with $P'.\sin\alpha'$ acting in the direction Ay.

Hence for any number of forces P, P'...... making angles α, α' ...... with Ax

all the forces are equivalent to

$P.\cos\alpha + P'.\cos\alpha' + ...$ in the direction Ax

together with $P.\sin\alpha + P'.\sin\alpha' + ...$ in the direction Ay.

For shortness' sake let $P.\cos\alpha + P'.\cos\alpha' + ... = X$,

and $P.\sin\alpha + P'.\sin\alpha' + ... = Y$.

Also let R, the resultant of all the forces, make an angle θ with Ax.

Then R is equivalent to $R.\cos\theta$ acting in the direction Ax,

together with $R.\sin\theta$ acting in the direction Ay,

$$\therefore \quad R.\cos\theta = X,$$

$$R.\sin\theta = Y.$$

From which we obtain

$R^2 = X^2 + Y^2$, which gives the magnitude of the resultant,

$\tan\theta = \frac{Y}{X}$, which gives the direction of the resultant.

XXX. *To find the conditions of equilibrium of a system of forces acting in one plane at a point.*

We have seen that the Resultant of any number of forces $P, P'\ldots$ may be determined in magnitude from the equation

$$R^2 = X^2 + Y^2,$$

where $$X = P.\cos\alpha + P'.\cos\alpha' + \ldots\ldots\ldots$$

and $$Y = P.\sin\alpha + P'.\sin\alpha' + \ldots\ldots\ldots$$

Now in order that $P, P'\ldots$ may be in equilibrium, their resultant must be zero:

that is, $$R = 0,$$

$$\therefore\ X^2 + Y^2 = 0.$$

But as the left-hand member of this equation consists of two terms which, being squares, are essentially *positive*, their sum cannot be equal to 0 unless each be separately equal to 0;

that is, $$X^2 = 0, \text{ and } Y^2 = 0,$$

and therefore $$X = 0, \text{ and } Y = 0,$$

$$\therefore\ P.\cos\alpha + P'.\cos\alpha' + \&c. = 0,$$

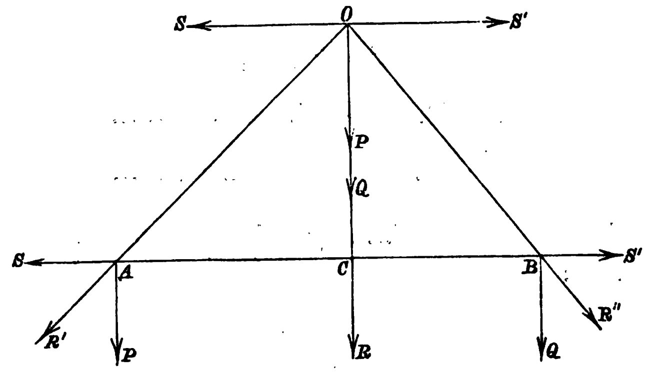
and $$P.\sin\alpha + P'.\sin\alpha' + \&c. = 0.$$

These are the conditions of equilibrium, which may be expressed in words thus:

"The sum of the forces resolved *in any two directions at right angles to each other* must be severally *zero*."

31. *To find the resultant of two forces whose directions are parallel.*

Case I. When the forces act towards the same parts.



Let A, B be any two points in the lines of action of the two forces P, Q, acting in the parallel directions AP, BQ.

At A apply any force S in the direction BAS, and at B apply an equal force S' in the direction ABS': this will not affect the combined action of the other forces.

Now S, P acting at A are equivalent to a single force R' acting in some direction AR'; and S', Q acting at B are equivalent to a single force R'' acting in some direction BR''. Let these two pairs of forces be replaced by R', R'', whose directions will meet in some point O: and let the points of application of R', R'' be transferred to O.

Draw OCR parallel to AP and BQ, and SOS' parallel to AB.

Now let R' acting at O be resolved into two components in directions OS and OC, which will clearly be S and P; and let R'', acting at O, be resolved into two components in directions OS' and OC, which will clearly be S' and Q.

Then S and S', being equal and opposite, will balance each other, and may therefore be removed, and there will remain P and Q acting at O in the line OCR.

Hence if R be the resultant of P and Q,

$$R = P + Q.$$

Again, in the triangle ACO, the sides are proportional to S, P, R'; and in the triangle BCO, the sides are proportional to S', Q, R'';

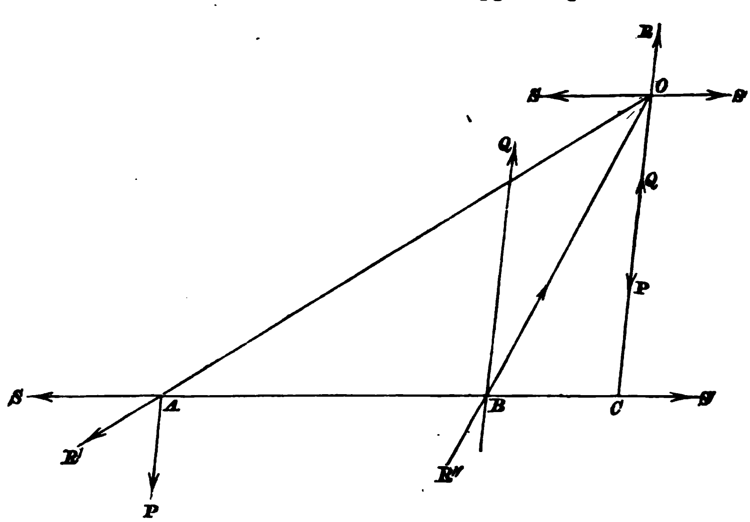
$$\therefore \frac{P}{S}=\frac{OC}{AO} \text{ and } \frac{S'}{Q}=\frac{BC}{OC};$$

$$\therefore \frac{P}{S}\times\frac{S'}{Q}=\frac{OC}{AC}\times\frac{BC}{OC};$$

$$\therefore \frac{P}{Q}=\frac{BC}{AC}.$$

Hence the resultant passes through a point C which divides AB into segments inversely proportional to the forces.

Case II. When the forces act towards opposite parts.



Let A and B be any two points in the lines of action of the two forces P, Q which act in the directions AP, BQ.

At A apply any force S in the direction BAS, and at B apply a force $S'=S$ in the direction ABS'. Then S and P acting at A have a resultant R'; and S' and Q acting at B have a resultant R''.

Let the directions of R' and R'' meet in O. Draw SOS' parallel to AB.

At O resolve the force R' into two components, S acting along OS and P acting along ROC which is parallel to AP and BQ.

Also resolve R'' into two components, S' acting along OS', and Q acting along COR.

Then S and S' will balance each other, and may be removed, and if R be the resultant of P and Q, there will remain a force acting along COR, such that

$$R = Q - P.$$

Also since the sides of the triangle ACO are proportional to P, R', S,

$$\frac{P}{S} = \frac{OC}{AC},$$

and since the sides of the triangle BCO are proportional to Q, R'', S',

$$\frac{S'}{Q} = \frac{BC}{OC};$$

$$\therefore \frac{P}{S} \times \frac{S'}{Q} = \frac{OC}{AC} \times \frac{BC}{OC}, \text{ and } \therefore \frac{P}{Q} = \frac{BC}{AC}.$$

32. *If three forces acting upon a rigid body balance each other, the lines in which they act must either be parallel or pass through a point.*

Fig. I.

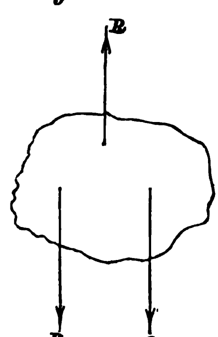
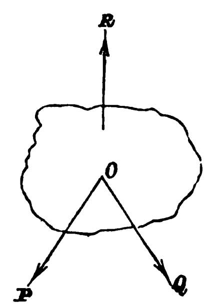


Fig. II.



Let P, Q, R be the forces.

First, suppose P and Q to be parallel, (as in fig. I.)

Then they will have some resultant acting in a direction parallel to each of them. But this force, since it counteracts R, must be in direction exactly opposite to the direction in which R acts.

Consequently the line in which R acts must be *parallel* to the directions of P and Q.

Next, suppose the lines of direction of P and Q to meet in a point O, (as in fig. II.)

Then the resultant of P and Q will pass through the point O.

But this force, since it counteracts R, must be in direction exactly opposite to the direction in which R acts.

Consequently the line in which R acts must *pass through the point* O.

EXAMPLES ON PART I.

1. If an ounce were taken as the unit of weight, how would a ton be represented?

2. If 1lb. be represented by a line 1 inch long, how will a ton be represented?

3. If an ounce were represented by an inch, what would a mile represent?

4. A cubic foot of a substance weighs 4 cwt.: what volume of another substance 5 times as dense will weigh 7 cwt.?

5. The magnitudes of two bodies are as three to two, the weights as two to one; compare the densities.

6. The magnitudes of three bodies are as 2 : 3 : 4, and their weights as 4 : 3 : 2; compare their densities.

7. Apply forces of 1, 2, 5 and 7lbs. respectively to a point so as to give the smallest possible resultant, the forces all acting in the same straight line.

8. Apply forces of 3, 7 and 10lbs. to a point so as to keep it at rest.

9. Place three forces which are in the proportion of 3, 4 and 5 so that they may keep a point at rest.

10. If two forces, acting at right angles to each other, be in the proportion of $1 : \sqrt{3}$, and their resultant be 10lbs., find the forces.

11. Given the direction and magnitude of the resultant, determine the directions of the component forces, when they are each equal to the resultant.

12. If the component forces be inclined at 120°, and the resultant be perpendicular to one of them, compare the forces.

13. If the forces be P and $2P$, and the angle between them four thirds of a right angle, determine the magnitude of the resultant.

14. Three forces whose magnitudes are 6, 8 and 10lbs. respectively, acting upon a point, keep it at rest, prove that the directions of two of the forces are at right angles to each other.

15. What will be the direction of the resultant (1) if one of the components be twice as great as the other, and the angle between their directions 120 degrees, (2) if the components be equal, and one act due East, and the other North-West?

16. P and Q are two forces applied to a point in directions at right angles to one another; P is 90lbs., Q is 120lbs.; find the direction and magnitude of their resultant.

17. P and Q are two forces applied to a point in directions at right angles to one another; P is 36lbs., Q is 48lbs.; what force must be applied and in what direction to produce equilibrium?

18. If two forces be inclined to each other at an angle of 135°, find the ratio between them when the resultant is equal to the smaller force.

19. A string passing round a smooth peg is pulled at each extremity with a force equal to the strain on the peg; find the angle between the directions of the two portions of the string.

20. Two strings at right angles to each other support a weight, and one string makes an angle of 30° with the vertical line. Compare the tensions of the strings.

21. If AB, AC represent two forces, and D is the middle point of BC, then the resultant will act along AD, and its magnitude will be represented by $2AD$.

22. If three forces keep a point at rest, prove that the angle between the two greatest is larger than the angle between any other two.

23. If three equal forces acting upon a particle keep it at rest, show that their directions must be equally inclined to each other.

24. Show that if the angle at which two forces are inclined to each other be increased, their resultant is diminished.

25. The ends of a string are tied to the rings of a picture and the

string is then passed over a nail, from which the picture hangs. Show that the longer the string the less will be the tension. Will the pressure on the nail be affected by the length of the string?

26. A and B are fixed points on the circumference of a circle, P any other point on the circumference; show that if two constant forces act along PA and PB, their resultant will pass through one point, for all positions of P.

27. Two weights P and Q are joined together by a string and laid on the circumference of a vertical semicircle which is twice the length of the string. Find the position of equilibrium.

28. Show that if eight forces acting on a particle be represented in magnitude and direction by the straight lines drawn from the angular points of a quadrilateral to the middle points of the opposite sides, they will form a system in equilibrium.

29. Show that if four forces act at a point in the circumference of a circle, and be represented in magnitude and direction by the four straight lines drawn from that point to the angular points of a square inscribed in the circle, their resultant will be represented by four times the straight line drawn from the given point to the centre of the circle.

30. Through a point O within a parallelogram $ABCD$ straight lines POQ, MON are drawn parallel to the sides and meeting AB, BC, CD, DA in P, M, Q, N respectively: show that if three forces acting on a particle be represented by PM, NQ, CA, they will form a system in equilibrium.

31. A point is taken within or without a quadrilateral, and lines are drawn from it to the angular points of the quadrilateral; prove that the resultant of the forces represented by these lines is represented by four times the line joining this point and the point of intersection of the lines joining the middle points of the opposite sides.

ELEMENTARY STATICS.

PART II.

Of the Centre of Gravity.

1. THE attraction of the Earth on any body would, if unopposed, draw it towards the surface of the Earth.

The direction in which a particle would fall freely at any place is called the *vertical line* at that place.

A plane perpendicular to this vertical line is said to be *horizontal.*

The directions of the forces which the Earth exerts on the different particles composing a body are not, strictly speaking, parallel. But since the dimensions of any body we shall have to consider are very small compared with its distance from the centre of the Earth, *we may consider these directions to be parallel.*

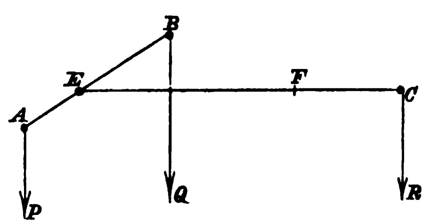
The resultant of this system of parallel forces is the weight of the body; and the point in the body at which this resultant acts is called *the Centre of Gravity of the body.*

We may suppose the whole weight of the body to be collected at the Centre of Gravity, and if it be in rigid connection with all the points in a body or a system of bodies, then the body or system would be in equilibrium in all positions, if the Centre of Gravity were supported.

Having thus explained the reasoning on which we proceed to investigate the nature of the Centre of Gravity of a body, we may give the following definition.

"The point at which the weight of a body or system may always be supposed to act is called the Centre of Gravity of the body or system."

2. *Every system of heavy particles has one and only one Centre of Gravity.*



Let A, B, C... be any number of heavy particles,

P, Q, R... the weights of A, B, C...

Suppose, first, that A and B are connected by a rigid rod without weight.

Now P and Q, being parallel forces acting in the same direction, are equivalent to a single resultant, the magnitude of which is $P+Q$, and which acts through a point E in the line AB, such that

$$P : Q :: BE : AE.$$

Hence, if E were supported, A and B would balance about E in any position.

E is then the centre of gravity of A and B, and the effect of P and Q will be the same as if A and B were collected into one particle and placed at E.

Now suppose $P+Q$ to act at E: then we may find the centre of gravity of $P+Q$ acting at E and R acting at C, as before, by taking a point F in the line EC, such that

$$P+Q : R :: CF : FE,$$

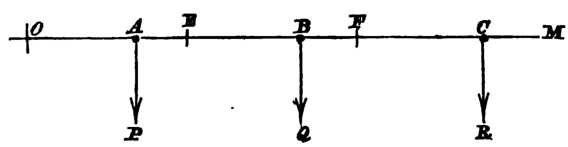
and we may suppose P, Q, R all collected at F: and so we may proceed for any number of particles.

Therefore every system of particles has a centre of gravity.

Also a system of particles can have *but one* centre of gravity.

For, if possible, let a system have two such points G and G', and let the system be turned about till the line joining G and G' is horizontal. Then we shall have the weight of the system acting in a vertical line through G, and also in another vertical line through G'; which is impossible, since it cannot act in two different lines at the same time.

3. *To find the centre of gravity of a number of particles lying in a straight line.*



Let A, B, C... be the several particles lying in the straight line OM:

P, Q, R... their weights.

Let O be a fixed point in the line.

In AB take a point E such that

$$P : Q :: BE : AE,$$

and therefore $$P.AE = Q.BE.$$

Then E is the centre of gravity of A and B, and $P+Q$ acting at E will produce the same effect as that produced by P acting at A and Q acting at B.

Now $(P+Q).OE = P.OE + Q.OE$

$$= P.(OA + AE) + Q.(OB - BE)$$

$$= P.OA + P.AE + Q.OB - Q.BE$$

$$= P.OA + Q.OB.$$

Next, to find the centre of gravity of $P+Q$ acting at E, and R acting at C.

Take a point F in EC such that

$$P+Q : R :: CF : EF,$$

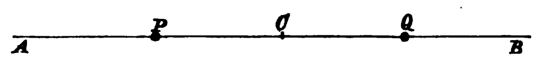
and therefore $(P+Q).EF = R.CF.$

Then F is the centre of gravity of A, B, C.

$$\begin{aligned}\text{Now } (P+Q+R).OF &= (P+Q).OF+R.OF\\ &= (P+Q).(OE+EF)+R.(OC-CF)\\ &= (P+Q).OE+(P+Q).EF+R.OC-R.CF\\ &= (P+Q).OE+R.OC\\ &= P.OA+Q.OB+R.OC.\end{aligned}$$

And so on for any number of particles.

4. *To find the centre of gravity of a right line.*



Let AB be the given right line.

We may regard AB as a line of equal particles uniformly arranged. We may then divide the line into a series of pairs of equal particles, each pair being equidistant from C, the middle point of AB.

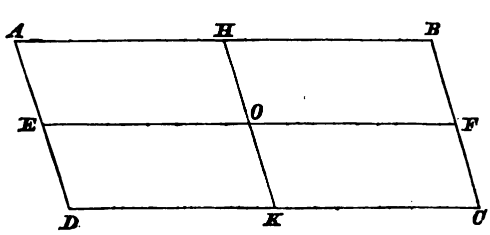
Let P and Q be such a pair of particles.

Then C will be the centre of gravity of P and Q.

Similarly each pair of the particles of which AB is composed will have C for its centre of gravity.

Therefore C will be the centre of gravity of the whole line AB.

5. *To find the centre of gravity of a parallelogram.*



Let $ABCD$ be a parallelogram, regarded as a uniform lamina of matter. Draw EF parallel to AB and CD, bisecting AD, BC in the points E, F; and HK parallel to AD and BC, bisecting AB, CD in the points H, K.

The point O in which HK, EF intersect is the centre of gravity of the parallelogram.

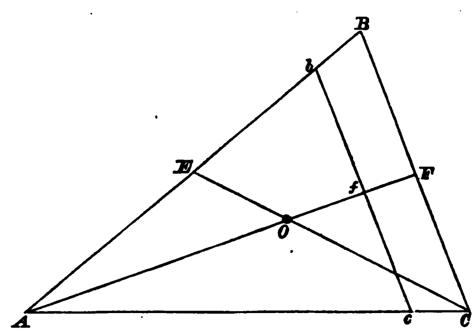
For by drawing lines parallel to BC and at equal distances from each other, we may divide the parallelogram AC into a number of equal small parallelograms whose lengths are all equal to BC, and breadths as small as we please: and we may take the breadths so small that each may be regarded as a line of particles, the centre of gravity of which is at its middle point, and which therefore is on the line EF, since EF bisects every line that is parallel to BC.

Hence the centre of gravity of the whole parallelogram lies in EF.

Similarly it may be shown to lie in HK.

Therefore O, the point of intersection of EF, HK, is the centre of gravity of the parallelogram.

6. *To find the centre of gravity of a plane triangle.*



Let ABC be a plane triangular lamina of matter.

We may suppose this triangle to be made up of a series of lines of particles running parallel to one of the sides, as BC.

Let bc be one of these lines.

Bisect BC in F and join AF, cutting bc in f.

Now

$$\begin{aligned} Af : fb &:: AF : FB \quad \text{(by similar triangles } Afb,\ AFB\text{)} \\ &:: AF : FC \quad \text{(since } FB = FC\text{)} \\ &:: Af : fc \quad \text{(by similar triangles } AFC,\ Afc\text{);} \\ &\therefore fb = fc. \end{aligned}$$

Similarly it may be shown that AF will bisect each of the lines parallel to BC, and hence the centre of gravity of each of the lines

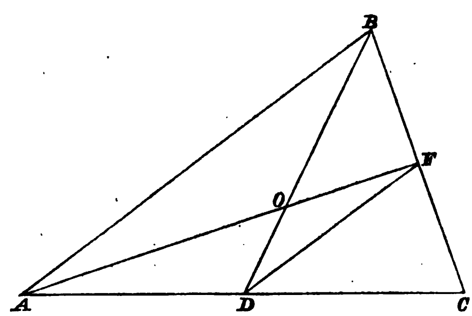
composing the triangle is in AF, and therefore the centre of gravity of the triangle is in AF.

Now bisect AB in E and join CE.

Then the centre of gravity of the triangle will be in CE.

Therefore the point O in which AF and CE cut each other will be the centre of gravity of the triangle.

7. *To show that if a line be drawn from any angle to the middle point of the opposite side, the centre of gravity of the triangle lies in this line at a distance from the angular point equal to two-thirds of the line.*



Draw BD and AF to the middle points of AC and BC.

Then O, the intersection of AF, BD is the centre of gravity of the triangle.

We have now to show that $BO = 2 . OD$.

Join FD.

Then, since FD bisects BC and AC it must be parallel to AB.

And since ABC, DFC are similar triangles,

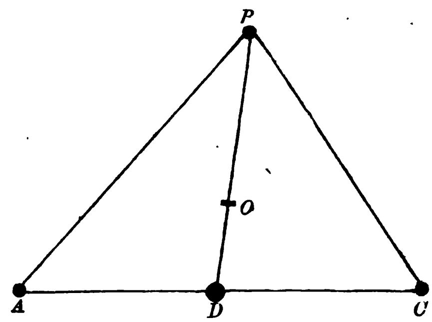
$$AB : DF :: AC : DC$$
$$:: 2 : 1.$$

Again, since AOB, FOD are similar triangles,

$$BO : OD :: AB : DF$$
$$:: 2 : 1;$$
$$\therefore\ BO = 2 . OD.$$

And hence BO is two-thirds of BD.

8. *The centre of gravity of a triangle coincides in position with the centre of gravity of three equal particles placed at the angular points.*



Let three particles, each of weight P, be placed at A, B, C.
Take D the middle point of AC.

Then D will be the centre of gravity of the particles acting at A and C, and we may suppose both to act at D.

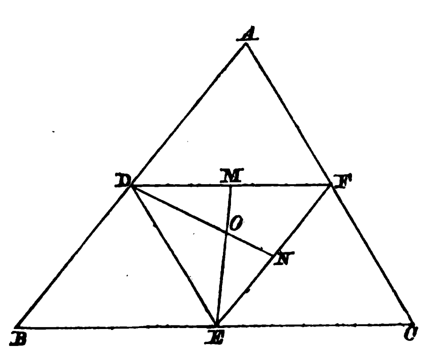
Then we have $2P$ acting at D and P at B, and the centre of gravity of these weights will be found by joining BD and taking in it a point O, such that

$$BO : OD :: 2P : P$$
$$:: 2 : 1,$$

i.e. O is the centre of gravity of the triangle.

9. *To find the centre of gravity of the perimeter of a triangle—regarding the sides as material lines of uniform thickness.*

Let D, E, F be the middle points of the sides of the proposed triangle ABC.



Then the centre of gravity of the perimeter ABC will be in the same position as the centre of gravity of three particles placed at D, E, F, whose weights are proportional to AB, BC, CA respectively.

Draw EM bisecting the angle DEF, and DN bisecting EDF.

Now $\quad DM : MF :: DE : EF$, by Euclid, VI. 3,

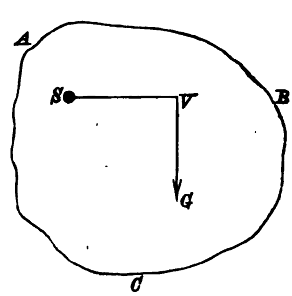
$:: AC : AB$, by similar triangles ABC, DEF.

Hence M is the centre of gravity of the two sides AB, AC; and therefore the centre of gravity of the whole perimeter lies in EM.

Similarly it lies in DN.

Therefore O the point of intersection of EM, DN is the centre of gravity required, and this by Euclid, IV. 4, is the centre of the circle inscribed in the triangle DEF.

10. *If a body be suspended from a point about which it can swing freely, it will rest with its centre of gravity in the vertical line which passes through the point of suspension.*



Let ABC be the body, G its centre of gravity, S the point of suspension.

Draw GV vertical and SV horizontal to meet GV in V.

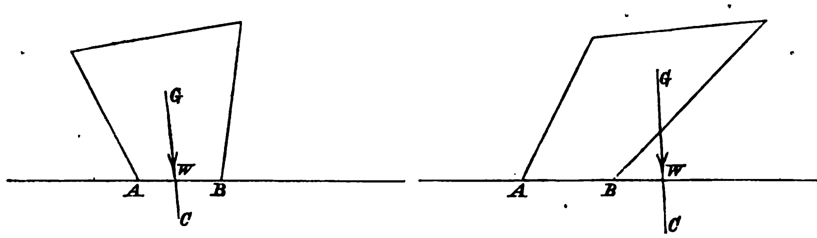
Then the only forces which act on the body are its weight, which acts in the vertical line VG, and the reaction arising from the fixed point S.

These two forces cannot balance each other unless they act in the same line in opposite directions, i. e., unless VG pass through S.

Therefore the body cannot be at rest unless the vertical line through G pass through S, and when this is the case the fixed point will exert a force on the body sufficient to balance the weight of the body and therefore equal and opposite to that weight.

11. *A body placed on a horizontal plane will stand or fall over, according as the vertical line drawn through the centre of gravity of the body falls within or without the base.*

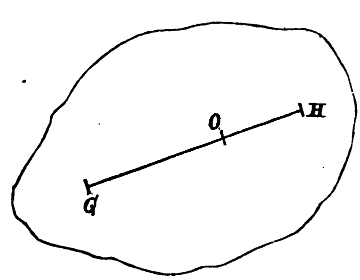
Fig. I. Fig. II.



Suppose the vertical line GC, passing through the centre of gravity G, to fall within the base, as in fig. I. Then we may suppose the weight of the body (W) to be concentrated at G. There will then be a vertical pressure of W downwards acting in the line GC, which will be counteracted by an equal and opposite pressure of the plane on which the body is placed acting upwards in the direction CG, and so equilibrium will be produced, and the body will stand.

But suppose, as in fig. II., that the line GC falls without the base: then there is no pressure equal and opposite to W, and the body will be twisted round B, the nearest point of contact in the base to the vertical line GC, and will fall.

12. *Having given the centre of gravity of a body and also the centre of gravity of a part of the body, to find the centre of gravity of the remainder.*



Let G and H be the centres of gravity of the two parts of the body, w and x the weights of the parts respectively.

Then if O be the centre of gravity of the whole body,

$$GO : OH :: x : w;$$

$$\therefore w \times GO = x \times OH;$$

$$\therefore OH = \frac{w \times GO}{x}.$$

Hence if G and O be given we can find H, by producing GO so far that the part produced $= \dfrac{w \times GO}{x}$.

13. *On stable and unstable equilibrium.*

(1) If a body under the action of any force be in a position of equilibrium, and a *very small* displacement be given to the body, if it then tend to return to the original position of equilibrium, that position is called one of *stable equilibrium.*

(2) If the body tend to move further from its original position, that position is called one of *unstable equilibrium.*

(3) If it remain in the new position which the displacement has given it, the position is said to be *neutral.*

Examples.

(1) A weight suspended by a string is an instance of stable equilibrium.

(2) A stick balanced on the finger is an instance of unstable equilibrium.

(3) A sphere resting on a horizontal table is an instance of neutral equilibrium.

14. *The position of the centre of gravity of a body may be sometimes determined by experiment in the following manner:*

Let the body be suspended from any point in its surface and let the line which is vertical and passes through the point of suspension be marked.

Then let the body be suspended from another point in its surface and let the line which is vertical and passes through the point of suspension be marked.

The point of intersection of the two lines in the body is the centre of gravity of the body.

EXAMPLES ON PART II.

1. A rod of 3 feet in length and 8 lbs. weight has a pound weight placed at one end; find the centre of gravity of the whole system.

2. A ball of weight 2 lbs. lies in the middle between two balls a foot apart of weights 4 lbs. and 1 lb.; find the centre of gravity of the three.

3. Find the centre of gravity of three equal heavy balls not in the same line.

4. Four equal particles are placed in a straight line, the distance between the first and second being one inch, between the second and third two inches, and between the third and fourth three inches; find their centre of gravity.

5. Show that if a number of triangles be described upon the same base and between the same parallels, their centres of gravity lie on a straight line.

6. Show that if the centre of gravity of three heavy particles placed at the angular points of a triangle coincides with the centre of gravity of the triangle, the particles must be of equal weight.

7. If the angular points of one triangle lie at the middle points of the sides of another, show that the triangles will have the same centre of gravity.

8. If two triangles are upon the same base, show that the line joining their centres of gravity is parallel to the line joining their vertices.

9. Two equal particles are placed on two opposite sides of a parallelogram; show that their centre of gravity will remain in the same position, if they move along the sides so as to be always equidistant from opposite angles.

10. Having given the positions of three particles A, B, C, and the position of the centres of gravity of A, B and A, C, find the position of the centre of gravity of B, C.

11. An equilateral triangle is inscribed in a circle; show that its centre of gravity will be the centre of the circle.

12. If the centre of gravity of a triangle inscribed in a circle coincide with the centre of the circle, show that the triangle is equilateral.

13. From a given square show how to cut a triangle having one side of the square for its base, so that the centre of gravity of the remaining portion may be at the vertex of the triangle.

14. Why does a man when he is carrying a weight with one arm extend the other?

15. A uniform flat rod whose length is 14 inches and weight 3 lbs. rests on a horizontal table; if a weight of 4 lbs. be placed on one end of the rod, find the greatest distance which the other end may be made to project beyond the table without the rod falling off.

16. AB the base of a square $ABCD$ is divided in E and the triangle CBE removed; show that the remainder will stand or fall according as BE bears to EA a less or greater ratio than $\sqrt{3} : 1$.

17. Find the centre of gravity of a trapezium of such form that the line joining one pair of its opposite angles divides it into two equal triangles.

18. Find the centre of gravity of four equal particles A, B, C, D, when the straight line AB bisects the straight line CD.

19. A heavy bar 14 feet long is bent into a right angle, the legs of which are 8 feet and 6 feet long respectively; prove that the distance of the centre of gravity of the bar so bent from the point in it which was its centre of gravity when it was straight is $\frac{9\sqrt{2}}{7}$ feet.

20. A triangle suspended from one of its angles has its base horizontal; show that the triangle is isosceles.

21. If a parallelogram be divided into four triangles by its diagonals, and the centres of gravity of the four triangles be joined, the joining lines will form another parallelogram.

22. If a triangle, right-angled at C and having the side opposite A double that opposite B, be suspended successively by A and C from a peg in a vertical wall, find the angle between the two positions of AC.

23. A right-angled triangle, whose acute angles are to each other as 1 : 5, is suspended from the right angle; determine the inclination of the hypothenuse to the vertical.

24. An isosceles triangle is suspended successively from the angular points of its base; show that the two positions of the base will be at right angles, if the base of the triangle be two-thirds of its altitude.

ELEMENTARY STATICS.

PART III.

Of Moments.

1. We have hitherto treated chiefly of the tendency of forces to produce motion of a particle or body *away from* a fixed point, i.e., to produce what is termed *displacement by translation.*

We shall now have to consider also the tendency of forces to produce motion *round* a fixed point, i.e., to produce what is termed *displacement by rotation.*

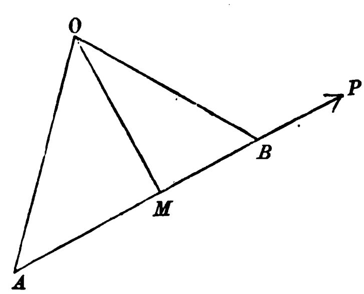
2. The MOMENT of a force about a point is defined to be *the product of the force into the perpendicular distance of its line of action from that point.*

The moment of a force about a point in its own line of action is consequently zero.

We shall explain hereafter the manner in which the moment of a force *measures* the effect of the force as a *physical* agency, and we shall first explain certain *geometrical* relations existing between the moments of component and resultant forces, which can be easily deduced from a *geometrical* interpretation of the definition given above.

3. *Geometrically* we can represent a force by a line containing as many units of length as the line contains units of force, and consequently we can represent a moment *geometrically* by a rectangle, contained by the lines representing the force and the perpendicular distance.

4. Let us now suppose that a force P represented in magnitude and direction by the line AB acts at A, a point in a straight rod AO moveable about O.



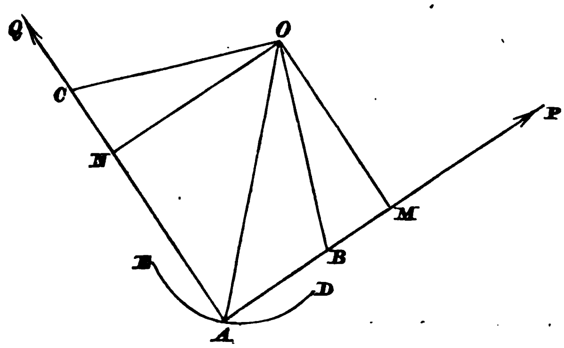
Draw OM at right angles to the line of action of P, and join OB.

Then, according to our definition, the moment of P about O will be represented *geometrically* by the rectangle contained by AB, OM.

Now the rectangle contained by AB, OM is double of the triangle AOB, and thus the triangle AOB will represent the half-moment of P about O.

We conclude, then, *that the moment of a force may be represented geometrically by twice the area of the triangle whose vertex is the point, and whose base is a line representing the force in magnitude and direction.*

5. Let us now take the case of two forces P and Q, represented by the lines AB, AC, acting at A on the rod AO in such a way that the perpendicular drawn from the fixed point O to P's line of action falls *on one side* of AO and the perpendicular drawn from O to Q's line of action falls *on the other side* of AO.



The moment of P about O will then be represented by twice the triangle AOB, and the moment of Q about O will be represented by twice the triangle AOC.

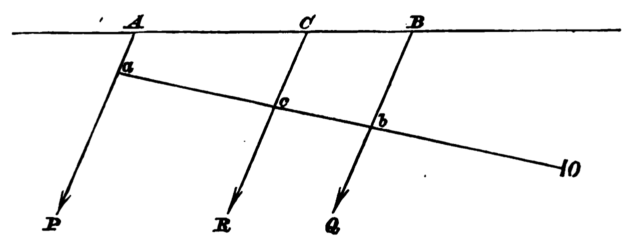
Now the force P tends to cause the point A to move along the circular arc AD and the force Q tends to cause the point A to move along the circular arc AE.

Thus the forces tend to twist the rod AO in contrary directions, and this difference we can express by the terms *positive* and *negative*. These terms are only relative and may be applied, at discretion, to express causes or effects that are directly opposed to each other, but for convenience sake we make the following statement:

The moment of a force may be considered negative or positive according as the force tends to twist the body in the same direction as the hands of a watch revolve, or the contrary.

6. *The algebraic sum of the moments of two parallel forces about any point in their plane is equal to the moment of their resultant about that point.*

Case I. When the forces act in *the same* direction.



Let A, B be two points in the lines of action of P, Q; and let C be the point in the line AB through which R, the resultant of P and Q, passes.

Take any point O.

Draw $Obca$ at right angles to the directions of the forces.

Then $\quad bc : ca :: BC : CA.$

Now $\quad P : Q :: BC : CA;$

$\therefore P : Q :: bc : ca;$

$\therefore P.ac = Q.bc.$

Also, sum of moments of P and Q round O

$$= P.Oa + Q.Ob$$

$$= P.(Oc + ac) + Q.(Oc - bc)$$

$$= P.Oc + P.ac + Q.Oc - Q.bc$$

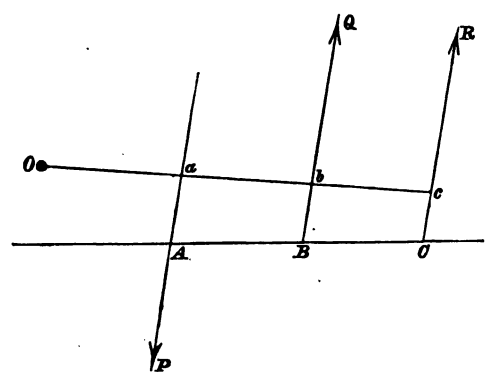
$$= P.Oc + Q.Oc$$

$$= (P + Q).Oc$$

$$= R.Oc$$

$$= \text{moment of } R \text{ round } O.$$

7. CASE II. When the forces act in *opposite* directions.



Let P and Q be the forces, of which Q is the greater.

Let A, B be two points in the lines of action of P, Q, and let C be the point in the line AB produced through which R, the resultant of P and Q passes.

Take any point O, and draw $Oabc$ at right angles to the directions of the forces.

Then $$bc : ac :: BC : AC.$$

Now $$P : Q :: BC : AC;$$

$$\therefore P : Q :: bc : ac;$$

$$\therefore P . ac = Q . bc.$$

Then, observing that the moment of P round O is *negative*, sum of moments of P and Q round O

$$= Q . Ob - P . Oa$$

$$= Q . (Oc - bc) - P . (Oc - ac)$$

$$= Q . Oc - Q . bc - P . Oc + P . ac$$

$$= Q . Oc - P . Oc$$

$$= (Q - P) . Oc$$

$$= R . Oc$$

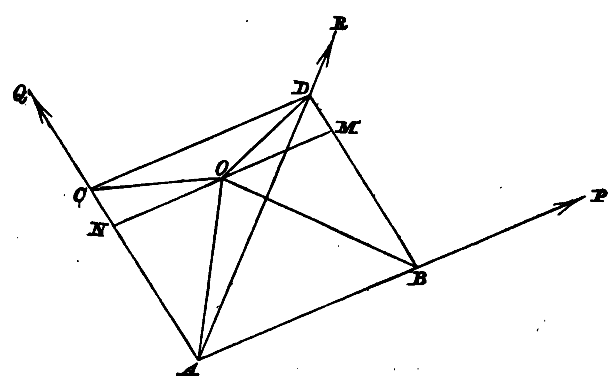
$$= \text{moment of } R \text{ round } O.$$

8. *The algebraic sum of the moments of two forces, meeting in a point and acting in one plane, about any point in the plane is equal to the moment of their resultant about that point.*

CASE I.

Let AB, AC represent the two forces P, Q.

Complete the parallelogram $ABDC$.



First, let O be a point *within* the angle BAC.

Draw MON parallel to AB and CD.

Now, as the figure is drawn, the moments of P and R about O are *positive*, and the moment of Q about O is *negative*, and we have to show that

$$2 \text{ triangle } AOB - 2 \text{ triangle } AOC = 2 \text{ triangle } AOD.$$

Now

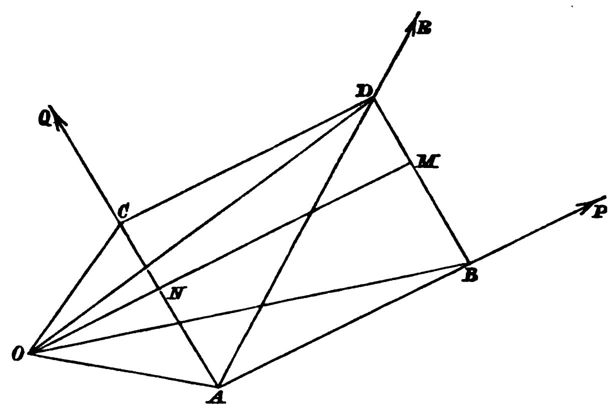
$$\text{parallelogram } BN = \text{parallelogram } BC - \text{parallelogram } MC;$$

$$\therefore 2 \text{ triangle } AOB = 2 \text{ triangle } ADC - 2 \text{ triangle } DOC$$
$$= 2 (\text{triangle } ADC - \text{triangle } DOC)$$
$$= 2 (\text{triangle } AOC + \text{triangle } AOD)$$
$$= 2 \text{ triangle } AOC + 2 \text{ triangle } AOD;$$

$$\therefore 2 \text{ triangle } AOB - 2 \text{ triangle } AOC = 2 \text{ triangle } AOD,$$

which proves the proposition.

Case II.



Next, let O be a point *without* the angle BAC.

Draw ONM parallel to CD and AB.

As the figure is drawn the moments of P, Q, R, about O are all *positive*, and we have to show that

$$2 \text{ triangle } AOB + 2 \text{ triangle } AOC = 2 \text{ triangle } AOD.$$

Now 2 triangle AOD

$$= 2\,(\text{quadrilateral } AOCD - \text{triangle } OCD)$$

$$= 2\,(\text{triangle } AOC + \text{triangle } ACD - \text{triangle } OCD)$$

$$= 2 \text{ triangle } AOC + 2 \text{ triangle } ACD - 2 \text{ triangle } OCD$$

$$= 2 \text{ triangle } AOC + \text{parallelogram } CB - \text{parallelogram } CM$$

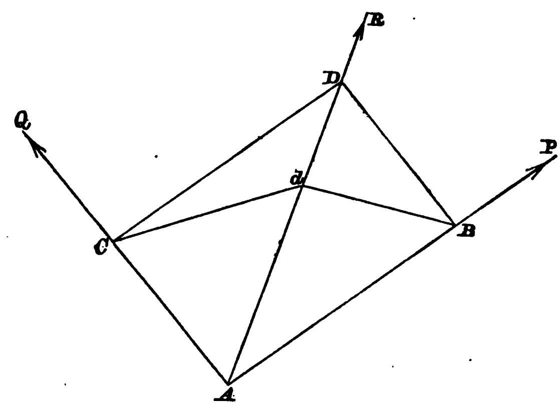
$$= 2 \text{ triangle } AOC + \text{parallelogram } BN$$

$$= 2 \text{ triangle } AOC + 2 \text{ triangle } AOB,$$

which proves the proposition.

From this and the two preceding articles we see that the algebraic sum of the moments of any two forces acting in one plane about any point in that plane is equal to the moment of their resultant about that point.

9. *The algebraic sum of the moments of two forces, acting in one plane, and meeting in a point, about any point in the line of action of the resultant is zero.*



Let AB, AC represent the two forces, P, Q.

Complete the parallelogram $ABDC$.

First to show that the sum of the moments of P and Q about the point D is zero.

$$\text{moment of } P \text{ about } D = 2 \text{ triangle } ADB,$$
$$\text{moment of } Q \text{ about } D = 2 \text{ triangle } ADC,$$

and the triangles ADB, ADC are equal;

$$\therefore \quad \text{moment of } P \text{ about } D = \text{moment of } Q \text{ about } D,$$

and the moments of P and Q are respectively *positive* and *negative;*

$$\therefore \quad \text{the sum of the moments of } P \text{ and } Q \text{ about } D \text{ is zero.}$$

Next let d be any point in the line of action of R.

Now $\qquad$ triangle AdB : triangle ADB :: Ad : AD,

and $\qquad$ triangle AdC : triangle ADC :: Ad : AD;

$\therefore$ triangle AdB : triangle ADB :: triangle AdC : triangle ADC;

and $\therefore$ since the triangles ADB and ADC are equal,

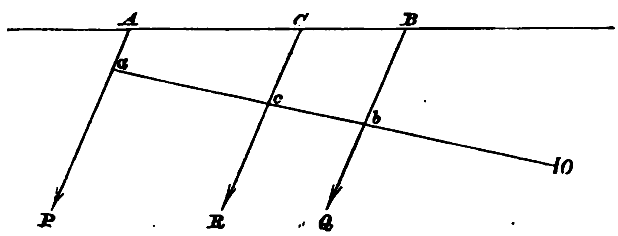
$$\text{triangle } AdB = \text{triangle } AdC;$$

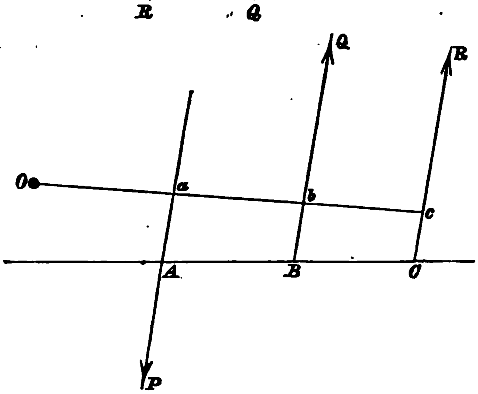
i.e. moment of P about d = moment of Q about d,

and the moments of P and Q are respectively positive and negative,

$\therefore$ the sum of the moments of P and Q about d is zero.

10. To show that the moments of two *parallel* forces about any point in the line of action of their resultant are equal in magnitude and opposite in direction we refer to the figures in articles 6 and 7, and taking moments of P and Q round any point c in the line of action of the resultant R, we observe





(1) Since $P \cdot ac = Q \cdot bc$,

the moments of P and Q are equal in magnitude.

(2) Since P and Q act in contrary directions with respect to their tendency to turn acb in Case I. and abc in Case II., regarded as rods moveable round the point c,

the moments of P and Q are opposite in direction.

11. We can readily extend the propositions proved in the preceding articles to any number of forces in one plane. For since the sum of the moments of two forces is equal to the moment of their resultant, we may substitute the resultant for the two forces; we may now combine this resultant with a third, and so on for any number of forces.

Hence we obtain the following conclusion:

The moment of the resultant of any number of forces in one plane, taken with respect to any point in that plane, is equal to the algebraic sum of the moments of the several forces with respect to the same point.

ELEMENTARY STATICS.

PART IV.

Of Mechanical Instruments.

1. A MECHANICAL Instrument is a contrivance for making a force which is applied at one point available at some other point.

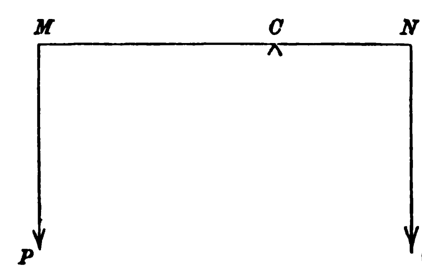
The Simplest Machines are rods used in pushing and ropes used in pulling, but what are called The Simple Machines, or Mechanical Powers, are

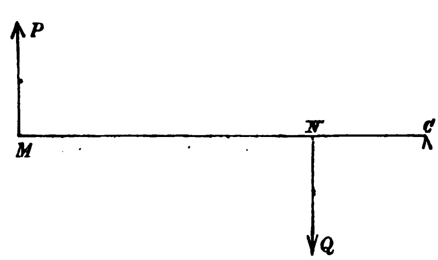
1. The Lever.
2. The Wheel and Axle.
3. The Pulley.
4. The Inclined Plane.
5. The Screw.
6. The Wedge.

THE LEVER.

2. A rigid rod capable of turning round a fixed point in the rod is called a *Lever*. The point about which it can turn is called the *Fulcrum*, and the parts into which the rod is divided by the fulcrum are called the *arms* of the lever. When the arms are in a straight line, the machine is called a *straight* lever: in all other cases it is called a *bent* lever.

3. *If two forces acting at right angles on a straight lever produce equilibrium the moments of the forces about the fulcrum are equal.*





Let P and Q be the forces acting at the points M, N, and balancing each other round the fulcrum C.

The force which the fulcrum exerts upon the lever must be equal and opposite to the resultant of the forces P and Q.

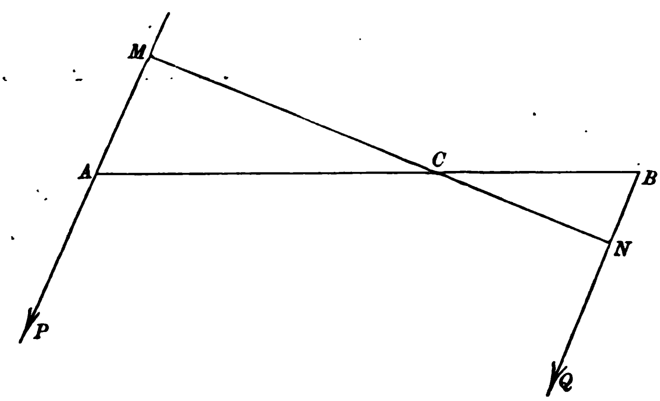
Then, since C is the point through which the resultant of P and Q passes, it follows from Art. 31 of Part I. that

$$\frac{P}{Q} = \frac{CN}{CM},$$

$$\therefore P \cdot CM = Q \cdot CN,$$

which is the property required.

4. *If two* parallel *forces acting at the extremities of a lever and tending to twist the lever opposite ways produce equilibrium, the moments of the forces about the fulcrum are equal.*



Let P and Q be two parallel forces balancing each other on the lever AB round the fulcrum C.

The force arising from the reaction of the fixed point C must be equal and opposite to the resultant of the forces P and Q.

Then we can show, as in Art. 31 of Part I., that

$$\frac{P}{Q}=\frac{BC}{AC}.$$

Now draw MCN at right angles to the directions of P and Q.

Then by similar triangles

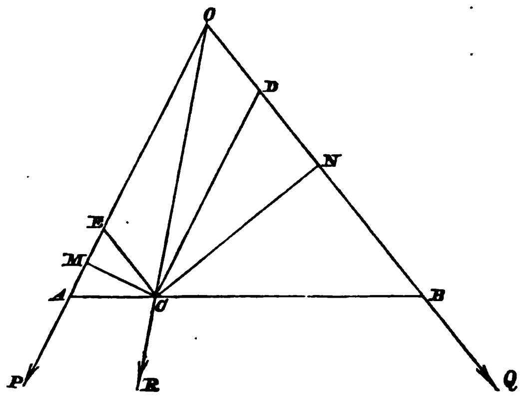
$$\frac{BC}{AC}=\frac{CN}{CM},$$

$$\therefore \frac{P}{Q}=\frac{CN}{CM},$$

$$\text{or} \quad P \times CM = Q \times CN,$$

which is the property required.

5. *If two forces which* are not parallel *acting at the extremities of a lever and tending to twist the lever opposite ways produce equilibrium, the moments of the forces about the fulcrum are equal.*



Let P and Q be the two forces acting at the ends of the lever AB.

Produce the lines of direction of P and Q to meet in O.

Then P and Q may be supposed to act at O, and we may obtain their resultant OR by the parallelogram of forces.

It is evident that this resultant must pass through the fulcrum C, for a single force could in no other way keep the lever at rest.

Draw CD parallel to OP and CE parallel to OQ, and CM and CN at right angles to OP and OQ respectively.

Then the sides of the triangle COD being parallel to the directions of the three forces P, Q, R may be taken to represent P, Q, R in magnitude.

Then $$\frac{P}{Q} = \frac{CD}{OD} = \frac{CD}{CE}.$$

Now the triangles CME, CND are similar, since the right angles CME, CND are equal, and angle CEM = angle DOE = angle CDN.

Hence $$\frac{CD}{CE} = \frac{CN}{CM},$$

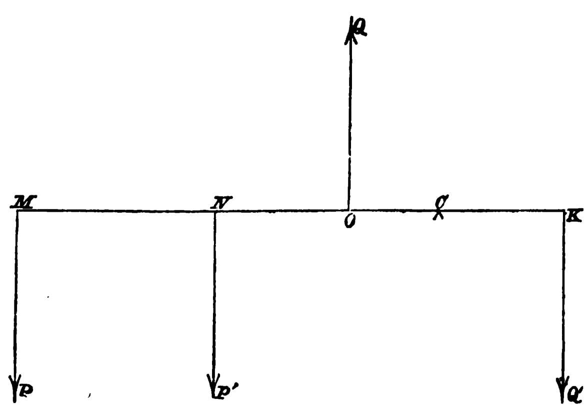
$$\therefore \frac{P}{Q} = \frac{CN}{CM},$$

and $$\therefore P \times CM = Q \times CN,$$

which is the property required.

The theorems established in the three preceding articles admit of a simpler demonstration by assuming the principle of moments, but it seems desirable to prove them by an independent process.

6. If more than two forces act on a lever in one plane and balance about the fulcrum C, the resultant of the forces must pass through C, and the algebraic sum of the moments of the forces about C must be zero; or, in other words, the sum of the moments of the forces which tend to turn the lever in one direction about C, must be equal to the sum of the moments which tend to turn the lever in the contrary direction.



Thus if the forces P, P', Q, Q', acting at right angles to the straight lever MK in the directions indicated in the diagram, are in equilibrium round the fulcrum C,

$$P \,.\, CM + P' \,.\, CN = Q \,.\, OC + Q' \,.\, CK.$$

7. We can now explain what is meant by the Moment of a Force about a point, regarded as a *physical* agency.

The unit of moment is the product of a unit of force into a unit of distance.

The physical effect of a force to produce rotation about a point will consequently be *measured* by the product of the *number* of units of force it contains into the *number* of units of distance contained in its perpendicular distance from the given point.

Thus if the unit of force be 1 lb., and the unit of distance 1 inch, the moment about a point of a force 5 lbs. acting at a perpendicular distance of 4 inches from the given point will be measured by the number 20.

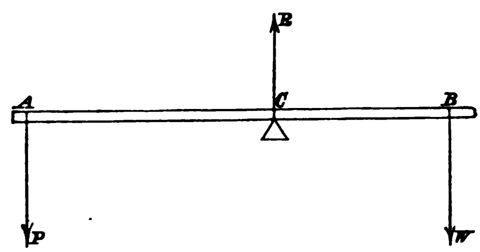
The usefulness of this numerical representation of a moment is apparent from the consideration that by means of it we can readily compare the effects of two forces which tend to cause rotation in contrary directions.

Thus if we have two forces of 2 and 5 lbs. acting at perpendicular distances of 10 and 4 inches respectively from the fulcrum of a lever, and tending to turn the lever in opposite directions, we conclude that no motion will ensue, because the product of the numbers 2 and 10 is equal to the product of the numbers 5 and 4.

Hence we see that although in discussing *geometrically* the relation of forces it is convenient to represent moments by *areas*, when we are treating of the *physical* effect of forces it is convenient to represent moments by *numbers*.

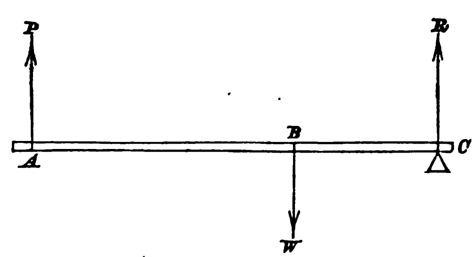
8. Levers are sometimes divided into three classes according to the relative position of the points where the *power* and *weight* are applied with respect to the fulcrum.

In levers of the *first class*, the power and weight are applied on *opposite* sides of the fulcrum C, and act in the same direction.



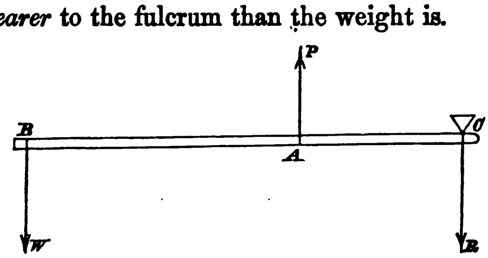
Examples. A poker between the bars of a grate, raising the coals. A pair of scissors is a double lever of this class.

In levers of the *second class*, the power and weight are applied on the *same side* of the fulcrum, and act in opposite directions, the power being applied at a *greater* distance from the fulcrum than the weight is.



Examples. An oar, the blade of the oar in the water being the fulcrum. A pair of nutcrackers is a double lever of this class.

In levers of the *third class*, the power and weight are applied on the *same side* of the fulcrum, and act in opposite directions, the power being *nearer* to the fulcrum than the weight is.



Examples. A man lifting a long ladder with one end resting on the ground. A pair of tongs is a double lever of this class.

9. Conditions of Equilibrium of a Lever.

I. When the lever is a *straight* one and the power and weight act *perpendicularly* to the arms, as in any of the three cases represented on the preceding page.

Let R be the force, or reaction, which the fulcrum exerts upon the lever, and the lever upon the fulcrum in the opposite direction, then the lever is kept at rest by the three forces P, W, R, and these forces must satisfy the conditions of equilibrium of three forces.

Hence, since the directions of P and W are parallel, R must also act in a parallel direction, and in

fig. 1. $R = P + W$,

fig. 2. $R = W - P$,

fig. 3. $R = P - W$.

Also the moments of any two of the forces P, W, R about a point in the line of action of the third must be equal in magnitude and of opposite tendency.

Hence, taking the moments of P and W about C, we have in *each of the three cases*

$$P \times AC = W \times BC.$$

II. When the lever is of *any form* and the power and weight act in any given directions, as in Art. 5, page 55.

In this case taking moments about the fulcrum, $P \times$ the perpendicular on P's direction $= W \times$ the perpendicular on W's direction.

The pressure on the fulcrum is determined by Art. 25 of Part I., for if θ be the angle between the directions of P and W,

$$R^2 = P^2 + W^2 + 2PW \,.\, \cos\theta.$$

10. The force applied to a machine to set it in motion is called the Power (P), and the resistance to be overcome is called the Weight (W).

In the propositions which we are now discussing we determine the value of the Power which would suffice to balance the weight, and any increase in this value of P will of course enable us to work the machine.

The *efficiency* or *working power* of a machine will be measured by the fraction $\frac{W}{P}$.

When W is greater than P the machine is said to work at *a mechanical advantage*, and when W is less than P at *a mechanical disadvantage*.

To illustrate this from the cases of the Lever, we take each class of levers separately, and observe

(1) In a lever of the *first* class,

P will be *less* than W if AC be *greater* than BC.

P will be *greater* than W if AC be *less* than BC.

Hence, in this case,

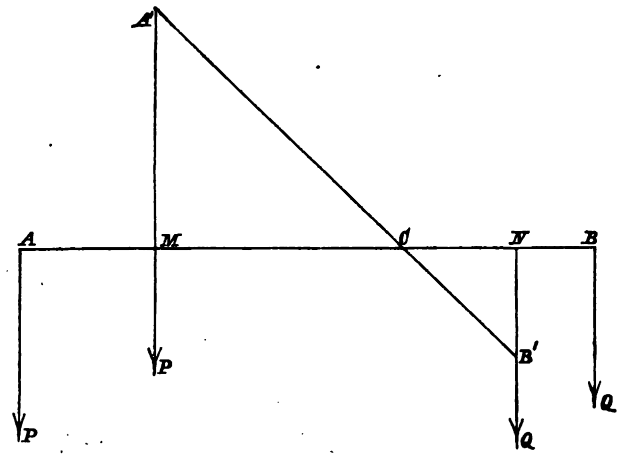
mechanical advantage is the result if P be further from the fulcrum than W,

mechanical disadvantage is the result if P be nearer to the fulcrum than W.

(2) In a lever of the *second* class P is always less than W. Hence in this case mechanical advantage is always gained.

(3) In a lever of the *third* class P is always greater than W. Hence in this case mechanical disadvantage is always the result.

11. *If two weights balance on a horizontal weightless lever, they will balance in every position of the lever.*



Let P and Q balance each other on the lever ACB when the lever is in a horizontal position.

Turn the lever round into the position $A'CB'$: there will still be equilibrium.

For, since P and Q hang *vertically*, their *lines of action* will cut the horizontal line AB at right angles at the points M, N.

Then since in the triangles $A'MC$, $B'NC$,

right angle $A'MC$ = right angle $B'NC$,

and vertical angle $A'CM$ = vertical angle $B'CN$:

$\therefore$ remaining angle $MA'C$ = remaining angle $NB'C$,

and so the triangles $A'MC$, $B'NC$ are equiangular and similar,

$$\therefore \quad CN : CB' :: CM : CA',$$

alternando $$CN : CM :: CB' : CA',$$

$$\therefore \quad CN : CM :: CB : CA.$$

But $$P : Q :: CB : CA,$$

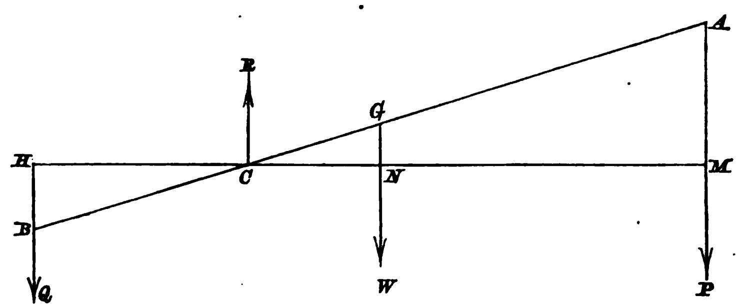
since P and Q balanced on the lever when horizontal,

$$\therefore \quad CN : CM :: P : Q.$$

Hence, since P and Q are inversely as the perpendiculars drawn from the fulcrum C to their lines of action they keep the lever at rest.

Q.E.D.

12. *If two weights balance each other on a straight heavy lever in any position which is not vertical, they will balance in any other position of the lever.*



Let P and Q be the two weights suspended from the points A, B of the lever whose fulcrum is C and centre of gravity G.

W = weight of the lever.

Draw HCM horizontal in the vertical plane in which the lever can move.

Suppose the lever to be inclined at any angle to the horizon, when in equilibrium.

Then, since P, Q, W act in vertical lines, the reaction R of the fulcrum must also be vertical, and we must have

$$R = P + Q + W.$$

Also taking moments about the fulcrum C,

$$P \,.\, CM + W \,.\, CN = Q \,.\, CH,$$

$$\therefore \quad P \,.\, AC \,.\, \frac{CM}{AC} + W \,.\, CG \,.\, \frac{CN}{CG} = Q \,.\, BC \,.\, \frac{CH}{BC}.$$

But by similar triangles,

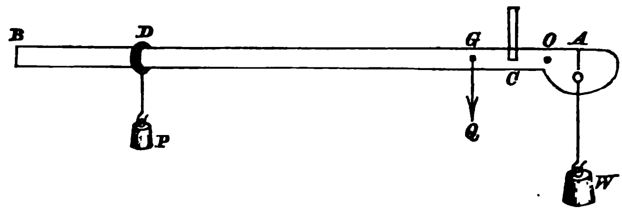
$$\frac{CM}{AC} = \frac{CN}{CG} = \frac{CH}{BC},$$

$$\therefore \quad P \,.\, AC + W \,.\, CG = Q \,.\, BC,$$

is the condition of equilibrium—and this is satisfied if the lever assume any other position inclined at any other angle to the horizon.

Hence the lever will balance in any other position.

THE COMMON OR ROMAN STEELYARD.



13. This balance consists of a straight lever AB suspended by the point C, and capable of turning round this point.

At the point A in the shorter arm is attached a hook, from which is suspended the substance whose weight W is required.

A ring D carrying a weight P of constant magnitude, can slide along the graduated arm CB, till P and W balance each other about C, when the lever is horizontal. The graduation at which P rests, when this is the case, indicates the weight of the substance. In graduating the arm BC, account must be taken of the weight of the lever: let Q be the weight of the lever and G its centre of gravity, D the point from which P is suspended when it balances W at A: then taking moments about C, we have

$$P \,.\, CD + Q \,.\, CG = W \,.\, CA \quad \text{.........................} (\alpha).$$

If on the arm CA we take a point O such that

$$P \,.\, CO = Q \,.\, CG,$$

the equation (α) becomes

$$P \,.\, CD + P \,.\, CO = W \,.\, CA,$$

or
$$P\,(CD + CO) = W \,.\, CA,$$

or
$$P \,.\, OD = W \,.\, CA;$$

$$\therefore \frac{OD}{CA} = \frac{W}{P}.$$

Now we may graduate OB by marking distances, measured from O, successively equal to CA, $2\,CA$, $3\,CA$, and so on.

When P rests at the first of these graduations,

$$OD = CA \text{ and } \therefore W = P.$$

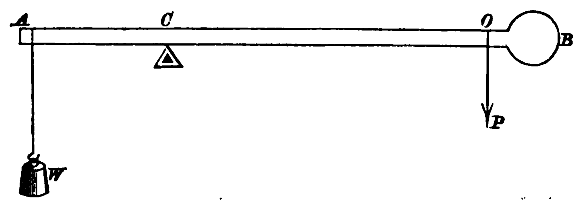
When P rests at the second of these graduations,

$$OD = 2\,CA \text{ and } \therefore W = 2P,$$

and so on.

Thus when the weight of P is known, the weight of W is known also.

THE DANISH STEELYARD.



14. This instrument consists of a bar AB terminating in a ball B, which serves as the *power*, and the substance to be weighed is suspended from the end A; the fulcrum C being moved along AB till P and W balance.

To graduate the instrument.

Let P be the weight of the steelyard, acting at O the centre of gravity: and let C be the position of the fulcrum when P and W balance.

Taking moments about C,

$$W.AC = P.OC$$

$$= P.(AO - AC)$$

$$= P.AO - P.AC;$$

$$\therefore \; W.AC + P.AC = P.AO,$$

or,

$$(P + W).AC = P.AO;$$

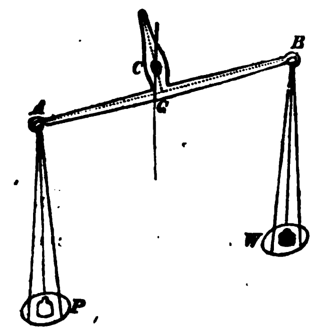
$$\therefore \; AC = \frac{P.AO}{P + W},$$

and by making W successively equal to P, $2P$, $3P$... we have

$$AC \text{ successively equal to } \frac{AO}{2}, \frac{AO}{3}, \frac{AO}{4} \ldots\ldots$$

and thus the graduations are determined.

THE COMMON BALANCE.



15. This balance consists of a lever AB called *the beam*, suspended from a fulcrum C about which it can turn freely; the point C is a little above the centre of gravity G of the beam, and from the extremities A, B of the arms GA, GB (which ought to be similar and equal) are suspended two scale-pans, in one of which is placed the substance whose weight W is required, and weights of known magnitude are placed in the other till their sum P just balances W; this being the case when the beam is exactly horizontal when in a position of rest. In this case if the arms are perfectly equal and similar, and the scale-pans also of equal weight, P will be exactly equal to W. If these weights differ by ever so little, the horizontality of the beam will be disturbed, and after oscillating for a short time, it will rest in a position inclined to the horizon, and the greater this inclination is for a given difference of P and W, the greater is the *sensibility* of the balance.

16. *The Requisites for a good Balance.*

(1) The beam ought to be horizontal when loaded with equal weights in the scales at A and B. This will be the case if the scales are of equal weight, and if the line drawn through C at right angles to AB divides the beam into two equal and similar arms.

(2) The balance ought to be *sensible*; i. e. the angle which the beam makes with the horizon ought to be easily perceptible when the weights P and W differ by a very small quantity.

(3) The balance ought to be *stable*; i.e. if the equilibrium be a little disturbed either way, there ought to be a decided and rapid tendency to return to its original position of rest, so as to ensure speed in the performance of a weighing.

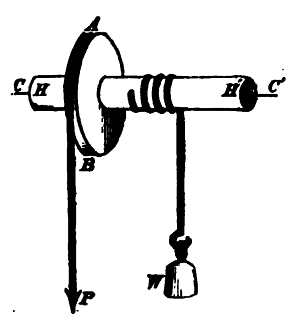
The comparative importance of these qualities in a balance will depend upon the service for which it is intended.

For weighing heavy goods, *stability* is of more importance.

For weighings requiring great accuracy, as in practical chemistry, *sensibility* is the quality desired.

A simple way of testing the accuracy of a balance is by interchanging P and W in the scales. The balance ought to retain the same position when this is done.

THE WHEEL AND AXLE.



17. This machine consists of a cylinder HH', called *the axle*, and a *wheel* AB, the two having a common axis terminating in pivots C and C', about which the machine can turn; the pivots resting in fixed sockets at C, C'.

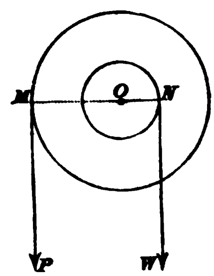
A rope, to one end of which the weight W is attached, passes round the axle, and has its other end fixed to the axle.

Another rope passes round the wheel, being attached at one end to the circumference of the wheel and at the other end the power P is applied.

The ropes pass round the *wheel* and *axle* in opposite directions, and thus tend to turn the machine in opposite directions.

The windlass and capstan are examples of the practical use of this mechanical instrument.

18. *To find the condition of equilibrium on the Wheel and Axle.*



Suppose the Wheel and Axle to be cut by a vertical plane at the point of their junction, and that this figure represents the section.

We may then suppose P and W to act in this plane, and that they hang vertically touching the circles at M and N.

From O the common centre draw OM and ON.

Then OM and ON, being drawn from the centre of the circles to the points of contact, will be perpendicular to PM and WN.

The axis of the machine being at rest, we may consider the machine as a lever moveable round O as a fulcrum.

Then there will be equilibrium when

$$P : W :: ON : OM,$$

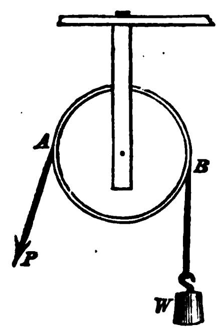
i.e. when $$P : W :: \text{radius of axle} : \text{radius of wheel}.$$

If the thickness of the ropes cannot be neglected, we must suppose P and W to act along the middle of the ropes, and in this case

$$ON = \text{radius of axle} + \text{radius of rope},$$

$$OM = \text{radius of wheel} + \text{radius of rope}.$$

THE PULLEY.



19. The *pulley* is a small circular disc or wheel having a uniform groove cut on its outer edge, and it can turn about an axis which passes through its centre. This axis rests in sockets within the *block* to which the pulley is attached.

When the block is fixed, the pulley is said to be fixed; in other cases it is moveable. A cord passes round the pulley along the groove, and at its extremities the power and weight are applied.

The pulley is very useful for changing the direction of a string; and as we shall suppose the groove to be perfectly smooth, the tension at all points of the string between the points of application of P and W will be the same.

When the pulley is fixed, no mechanical advantage is gained by its use beyond that of greater convenience in applying the force.

20. We may here conveniently give a more complete account of the *Tension of Strings*.

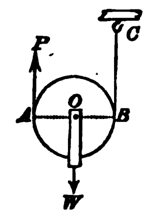
If we consider a string as a line of consecutive particles, when a force is applied at each end of the string each particle of the string is pulled in opposite directions by the forces which the consecutive particles on either side of it are compelled to exercise upon it. These forces are called *tensions*, and are the same at every particle of the string.

Suppose an engine attached to a truck by a coupling-chain to be just on the point of moving the truck. *Each link* of the chain is then acted upon by two equal and opposite forces, which act by means of the other links on either side of any particular link. The force with which the part of the chain on *one side* of any particular link resists the force exerted along the chain on the *other side* of the link is called the *tension* of the chain.

21. *To find the conditions of equilibrium on a single moveable Pulley.*

(1) When the strings are parallel.

Since the tension of the string $PABC$ which passes round the pulley is the same throughout, the tension *upwards* of the portions AP, BC are each equal to P: and since there is equilibrium, we may suppose the strings AP, BC *attached* to the pulley at A and B, the points where they quit the pulley; and the weight W, which is suspended from O, the axis of the pulley, is supported by the upward tension of the strings AP, BC.



Hence the resultant of the two tensions must be equal and opposite to W, and this resultant $= 2P$,

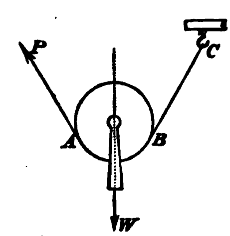
$$\therefore 2P = W$$

is the condition of equilibrium.

(ii) When the strings are not parallel.

Let the string quit the pulley at A and B.

Then since the tension along AP is equal to that along BC, their resultant will bisect the angle between them, and this resultant must be equal and opposite to the weight W suspended from the axis of the pulley, and acting in a vertical direction.



Hence AP, BC must be equally inclined to the vertical. Let θ be this inclination. Then the resultant of the two tensions, which we may regard as acting at A and B, is

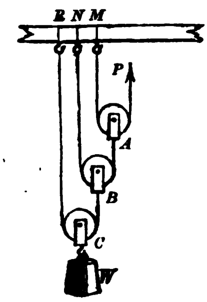
$$2P . \cos\theta,$$

and this must be equal to W,

$$\therefore 2P . \cos\theta = W$$

is the condition of equilibrium.

22. *To find the condition of equilibrium for a system of pulleys in which each pulley hangs by a separate string.*



In this system a string, acted on by the power P, passes round the pulley A, and is fastened to the block at M.

A string, attached to A, passes round the pulley B, and is fastened to the block at N, and so on for any number of pulleys.

The weight W is suspended from the lowest pulley.

Then, since W is supported by the tension of the strings RC, BC,

$$BC\text{'s tension} = \frac{W}{2}.$$

Again, $$\text{tension of } AB = \frac{\text{tension of } BC}{2} = \frac{W}{4},$$

and $$\text{tension of } PA = \frac{\text{tension of } AB}{2} = \frac{W}{8}.$$

But $$\text{tension of } PA = P,$$

$$\therefore P = \frac{W}{8}.$$

Thus, when there are three pulleys, $P = \dfrac{W}{2^3}$,

and similarly when there are n pulleys, $P = \dfrac{W}{2^n}$,

$$\therefore P : W :: 1 : 2^n.$$

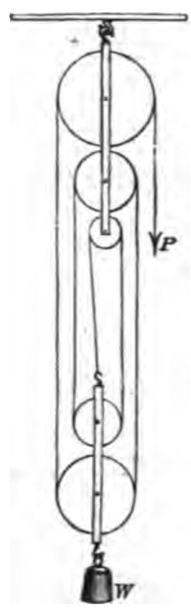
If the pulleys have weight, an additional force p will be required to assist P. Calling the weights of the pulleys, commencing with the *highest*, $w_1, w_2, w_3 \ldots\ldots w_n$,

$$p = \frac{w_1}{2} + \frac{w_2}{4} + \frac{w_3}{8} + \ldots\ldots \frac{w_n}{2^n},$$

the terms on the right-hand side of the equation being obtained by taking the formula $P=\frac{W}{2^n}$, and making $W=w_1, w_2 \ldots w_n$ successively, and $n=1, 2 \ldots n$, successively.

23. *To find the condition of equilibrium for a system of pulleys, where there are two blocks and the same string passes round the pulleys.*

In this system of pulleys the same string passes round each of the pulleys as in the figure.



Since the tension of the string is the same throughout, if n be the number of strings at the lower block, nP will be the *resultant* of the upward tensions of the strings upon the lower block.

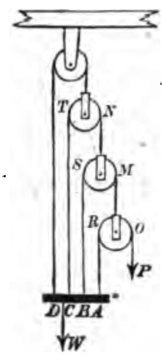
This resultant must be equal to W, when there is equilibrium,

that is, $$nP=W$$

is the condition required, which may be expressed thus,

$$P : W :: 1 : n.$$

24. *To find the condition of equilibrium in a system of Pulleys in which all the strings are attached to the weight.*



The figure represents the system.

The weight is supported by the tension of the strings RA, SB, TC,......

Now

tension of $RA = P$,

tension of $SB =$ tension of $MO =$ tension of $RA +$ tension of $OP = 2P$,

tension of $TC =$ tension of $NM =$ tension of $SB +$ tension of $MO = 2^2.P$,

and so on.

Therefore $\quad W = P + 2P + 2^2.P + \ldots\ldots$

and if there be n pulleys,

$$\begin{aligned} W &= P + 2P + 2^2.P + \ldots\ldots + 2^{n-1}.P \\ &= P(1 + 2 + 2^2 + \ldots\ldots + 2^{n-1}) \\ &= P.\left(\frac{2^n - 1}{2 - 1}\right) \\ &= P.(2^n - 1). \end{aligned}$$

If the pulleys have weight they *assist* P, and if we call the assistance, which they afford, p, and designate the weights of the pulleys beginning with the *lowest* by w_1, w_2, w_3......

$$p = (2^{n-1} - 1)\, w_1 + (2^{n-2} - 1)\, w_2 + \ldots\ldots + (2^2 - 1).\, w_{n-2} + (2 - 1).\, w_{n-1},$$

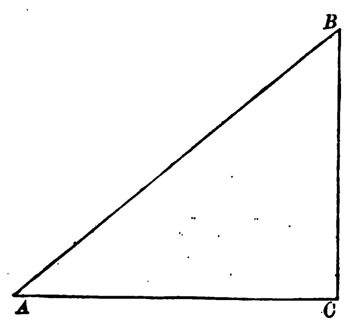
the terms on the right-hand member of the equation being obtained by taking the formula

$$W = P(2^n - 1),$$

and making P equal w_1, w_2......w_{n-2}, w_{n-1} successively, the number of pulleys being $n - 1$, $n - 2$ ...... 2, 1, successively.

THE INCLINED PLANE.

25. By an inclined plane, as a mechanical instrument, is meant a plane inclined to the horizon.



The figure represents a section of the inclined plane, made by a *vertical* plane perpendicular to the inclined plane.

AB is called *the length* of the plane.

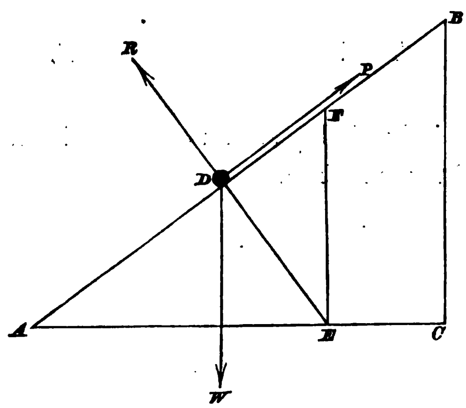
BC, which is taken to be perpendicular to AC, is called *the height* of the plane.

The angle BAC is called the *inclination* of the plane.

When a body is in contact with a smooth plane there is a mutual action between the body and the plane acting at right angles to the plane. The force thus brought to bear on the body is called the *reaction* of the plane, and the reason for this reaction being equal to the pressure of the body on the plane is to be explained thus:—

Reaction is always contrary and equal to action: or, the mutual actions of two bodies upon each other are always equal, and directed towards opposite parts. Whatever draws or presses another is as much drawn and pressed by that other. If any one presses a stone with his finger, his finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse will be equally drawn back towards the stone.

26. *To find the condition of equilibrium on a smooth Inclined Plane when the Power acts parallel to the plane.*



Let a body whose weight is W be pulled by a force P acting parallel to the plane, and let the body be at rest at the point D.

The body is acted upon by three forces:

P the power, acting parallel to AB,

W the weight, acting parallel to BC,

R the reaction of the plane acting at right angles to AB.

Produce RD to meet AC in E.

From E draw EF parallel to BC.

Then since the three sides of the triangle DFE are *parallel* to P, W, R respectively, the sides taken in proper order are also *proportional* to P, W, R;

$$\therefore \; P : W :: DF : FE.$$

Again, DFE, ABC are similar triangles, for

the right angle $FDE =$ the right angle BCA,

and the angle $DFE =$ the angle ABC,

and $\therefore$ the remaining angles FED, BAC are equal.

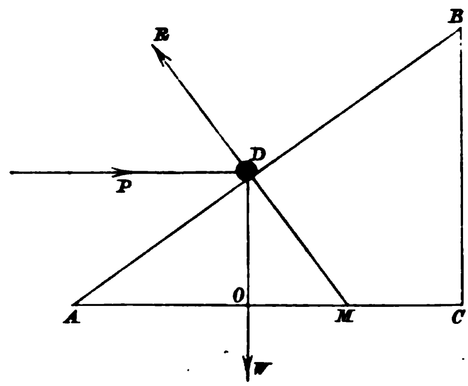
Hence $DF : FE :: BC : AB$,

$\therefore \; P : W :: BC : AB$

:: height of plane : length of plane,

which is the condition of equilibrium.

27. *To find the condition of equilibrium on the Inclined Plane when the Power acts horizontally.*



Let the body D be kept at rest by three forces:

P, the power, acting horizontally,

W, the weight of D, acting vertically downwards,

R, the resistance of the plane, acting at right angles to AB.

Produce RD to meet AC in M.

Then the sides of the triangle MOD are parallel, and therefore proportional, to P, W, R.

Now MOD, BAC are similar triangles, for

$$\text{angle } MOD = \text{angle } BCA,$$

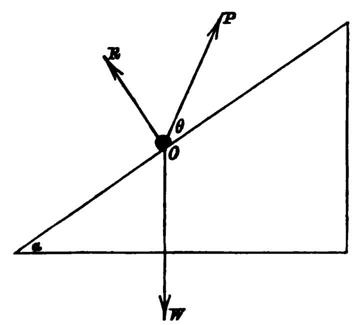
$$\begin{aligned}\text{and angle } MDO &= 90^\circ - \text{angle } ODA \\ &= \text{angle } BAC.\end{aligned}$$

$$\begin{aligned}\therefore\ P : W &:: OM : OD \\ &:: BC : AC.\end{aligned}$$

Also $$P : R :: BC : AB,$$

and $$W : R :: AC : AB.$$

xxviii. By the aid of Trigonometry we can find the condition of equilibrium on an inclined plane in a more general form.



Let a body O, whose weight is W, be supported on a smooth inclined plane by a force P, the direction of which makes an angle θ with the plane.

Let the figure represent a section of the inclined plane, made by a vertical plane perpendicular to the inclined plane.

Let α be the inclination of the plane.

Then the forces acting on the body O are

W, the weight of the body, acting vertically downwards,

R, the reaction of the plane, acting at right angles to the plane,

P, the given force.

Then, since there is equilibrium, we have (see page 16)

$$P : W : R :: \sin ROW : \sin ROP : \sin POW,$$

$$:: \sin(180^\circ - \alpha) : \sin(90^\circ - \theta) : \sin(90^\circ + \alpha + \theta),$$

$$:: \sin\alpha : \cos\theta : \cos(\alpha + \theta).$$

Two particular cases are to be especially noticed:

(1) When P acts *parallel* to the plane, $\theta = 0$, $\cos\theta = 1$,

$$\text{and } \therefore P : W : R :: \sin\alpha : 1 : \cos\alpha.$$

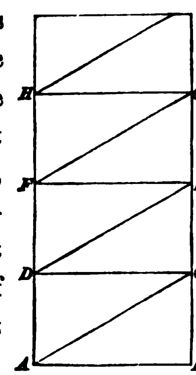
(2) When P acts *horizontally*, $\theta = -\alpha$, $\cos\theta = \cos\alpha$, $\cos(\alpha + \theta) = 1$,

$$\text{and } \therefore P : W : R :: \sin\alpha : \cos\alpha : 1.$$

THE SCREW.

29. The screw is a spiral thread running along the surface of a circular cylinder, which may be imagined to be generated thus:

Let AG be a rectangle whose base AB is exactly equal to the circumference of the cylinder; make the rectangles BD, CF, EH... equal in every respect, and draw the straight lines AC, DE, FG...; then if the rectangle BH be applied to the surface of the cylinder so that the base AB coincides with the base of the cylinder, the broken lines AC, DE, FG... will form a continuous line on the surface of the cylinder, the point C coinciding with D, E with F, and so on. If we now suppose this *line* to become a protuberant thread, perpendicular to the plane of the rectangle, we obtain a *screw*, in which the distance between any point of one thread and the one next below it, measured parallel to the axis of the cylinder, is everywhere the same and equal to BC.



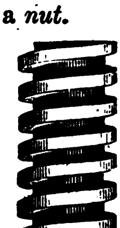
The angle CAB which the thread at any point makes with the base of the cylinder is called the *pitch* of the screw.

The screw formed on the *solid* cylinder, as above, works in a *hollow* cylinder of equal radius, in which a spiral groove is cut exactly equal and similar to the thread on the solid cylinder, and in which groove the thread of the solid screw can work freely.

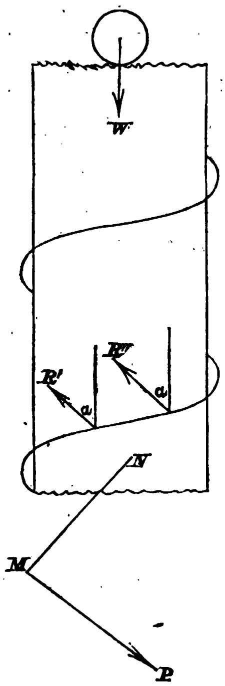
A solid and hollow screw related as above are called *companion screws;* and when in action, one of them is fixed and the other is turned by means of a lever fixed into the cylinder at right angles to its axis. By turning the lever a weight is raised, or a pressure produced, at the end of the screw, which pressure acts in direction of the axis of the screw.

When the solid screw is small, it is sometimes called a *nut.*

The annexed figure, representing the appearance of a solid screw, will assist the reader in understanding that a screw is nothing more than an inclined plane, constructed on the surface of a cylinder.



XXX. *To find the condition of equilibrium on the Screw.*



The forces acting on the screw will be

P, the power acting horizontally at right angles to the rod MN, whose length is a;

W, a weight placed at the end of the screw and acting vertically downwards,

and a series of forces R', R''... arising from the pressure of the hollow screw on each point of the solid screw with which it is in contact.

The forces R', R''... act at right angles to the thread and will therefore make an angle α, equal to the *pitch* of the screw, with the lines drawn vertically from the points of contact.

Resolving R', R''... vertically and horizontally, we shall have

$R'.\cos\alpha$, $R''.\cos\alpha$,... acting vertically upwards,

and $R'.\sin\alpha$, $R''.\sin\alpha$,... acting horizontally and tending to twist the screw in a direction contrary to that in which P tends to twist it.

Also each of these horizontal forces acts at an arm r, equal to the radius of the cylinder.

Hence $(R' + R'' + ...) . \sin\alpha . r = P . a$ ....(1),

and $(R' + R'' + ...) . \cos\alpha = W$ .... (2).

Dividing (1) by (2)

$$\tan\alpha \, . \, r = \frac{P \, . \, a}{W},$$

$$\text{or} \quad \frac{P}{W} = \frac{r \, . \tan\alpha}{a}.$$

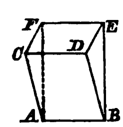
This condition of equilibrium may be expressed in another form, thus, since

$2\pi \, . \, r \, . \tan\alpha =$ distance between two threads, measured parallel to the axis,

and $2\pi \, . \, a =$ circumference of the circle described by M,

$$\frac{P}{W} = \frac{2\pi \, . \, r \, . \tan\alpha}{2\pi \, . \, a} = \frac{\text{distance between two threads}}{\text{circumference of circle whose radius is } MN}.$$

THE WEDGE.



31. The Wedge is a solid triangular prism.

Its two ends are equal and similar triangles.

Its three sides are rectangular parallelograms.

AB is called its *edge*: $CDEF$ its *head*.

It is used for separating bodies, or parts of the same body, which adhere strongly to each other.

The edge of the Wedge is introduced into a small cleft, and it is then driven forward by blows of a hammer applied at its head.

The mode of working this machine is quite different in principle from the method used in the other machines which have been described.

They are worked by the *regular* and *steady* application of a power, acting uniformly at that point of the machine to which it is applied, and *gradually* producing motion: but in this machine the power is applied by *sudden impulses*.

Hence any investigation for finding the relation between the power and weight in this machine must involve considerations, which cannot be explained by the principles of Elementary Statics.

Hatchets, chisels, nails, carpenters' planes, swords, are modifications of the wedge.

EXAMPLES ON PART IV.

LEVERS.

1. The force at the extremity of one arm of a straight lever of the first class is 12 lbs., the length of the arm is one foot and the pressure on the fulcrum is 16 lbs.: what is the length of the other arm of the lever?

2. If the weights on a lever are as 5 : 7, and the length of the lever is 36 inches, find the position of the fulcrum.

3. Weights equal to 7 lbs. and 11 lbs. balance each other when suspended from the ends of a lever whose length is 6 feet; find the position of the fulcrum.

4. If two weights of 2 lbs. and 5 lbs. balance on a lever whose weight is 2 lbs., compare the arms.

5. A heavy rod balances itself on a point one-third of its length from one end: if the rod be carried by two men, one at each end, what part of the weight will be supported by each?

6. Two men carry a uniform beam 6 feet in length and weighing 16 stone upon their shoulders, and at two feet from one end a weight of 4 stone is placed: what weight does each sustain, supposing the ends of the beam to rest on their shoulders?

7. Two men support a uniform heavy beam on their shoulders, which are at distances a and b from the ends; if the pressure on one man be r times that on the other, find the length of the beam.

8. The length of a horizontal lever is 12 feet, and the balancing weights at the ends are 3 lbs. and 6 lbs. respectively. How far ought the fulcrum to be moved for equilibrium if each weight be placed 2 feet from the ends of the lever?

9. How would the mechanical advantage of an oar be modified by lengthening that part of it which is within the rowlock?

10. A heavy uniform bar, 10 feet long and of given weight W, is laid over two props in the same horizontal line so that 1 foot of its length projects over one of the props. What must be the distance between the props that the pressure on one may be double that on the other?

11. If the weights on a lever be 8 lbs. and 7 lbs., and the arms 8 inches and 9 inches respectively, at what point must a force of 1 lb. be applied perpendicularly to the lever in order to keep them at rest?

12. Two weights P, Q, balance each other on a straight lever: if they be interchanged, determine the weight which must be added to or subtracted from either to produce equilibrium.

13. The arms of a lever are inclined to each other; shew that the lever will be in equilibrium with equal weights suspended from its extremities, if the point midway between the extremities be vertically above the fulcrum.

14. ABC is a weightless triangle having a right angle BAC, and $AB = 2 . AC$; if the triangle be suspended from A, and two weights P, Q, hanging at B, C, keep it at rest with the sides AB, AC equally inclined to the vertical, find the ratio of P to Q.

15. ACB is a weightless lever of which the arms CA, CB are straight and equal, and inclined to one another at an angle equal to a right angle and a half. When CA is horizontal, a weight P at A just sustains a weight W at B; and when CB is horizontal, W at B requires a weight Q at A to balance it; find the ratio of P to Q.

16. If the moveable weight for which a common steelyard is constructed be 1 lb., and a tradesman substitute a weight of 2 lbs.,

6

using the same graduations, show that he defrauds his customers, if the centre of gravity of the steelyard be in the longer arm, and himself, if it be in the shorter arm.

17. If the common steelyard consists of a uniform rod, whose weight is $\frac{1}{p}$ of the moveable weight, and the fulcrum be $\frac{1}{4}$ of the length of the rod from one end; show that the greatest weight that can be weighed is $\frac{3p+1}{p}$ times the moveable weight.

18. The arms of a balance are in the ratio of 19 : 20. The pan in which the weights are placed is suspended from the longer arm. What is the real weight of a body, which apparently weighs 38 lbs.?

19. A body, the weight of which is 1 lb., appears to weigh 14 ounces when it is placed in one scale of a false balance. What will be its apparent weight when placed in the other scale?

20. One pound is weighed at each end of a false balance and the sum of the apparent weights is $2\frac{1}{8}$ lbs., what is the ratio of the lengths of the arms?

21. If a balance be false, having its arms unequal and in the ratio of 15 : 16, find how much per lb. a customer really pays for tea which is sold to him from the longer arm at 3*s*. 9*d*. per lb.

WHEEL AND AXLE, AND PULLEYS.

1. If the radius of the wheel be 3 feet, the weight 18 lbs., and the power 3 lbs., what must be the radius of the axle?

2. In what way must the power act so that the pressure on the axle may be the least possible?

3. If the string to which the weight is attached be coiled in the usual manner round the axle, but the string by which the power is applied be nailed to a point in the rim of the wheel, find the position of equilibrium, the power and weight being equal.

4. Explain how the "capstan" possesses mechanical advantage, and if the radius of the axle be 2 feet, and 6 men push each with a force of 1 cwt. on spokes 5 feet long, find the weight they will just be able to support.

5. In a single moveable pulley if the strings be not parallel and $P = W$, what must be the angle between the strings?

6. Show that no mechanical advantage is gained by the single moveable pulley, unless the weight of the pulley be less than the power.

7. In a system of 3 pulleys a weight of 5 lbs. is attached to the lowest pulley. Supposing the weight of each pulley to be 3 lbs., find the force required to sustain equilibrium.

8. If there are 4 pulleys, whose weights, commencing with the highest, are 1, 2, 4, and 8 lbs. respectively, and W is 160 lbs., find P.

9. If there be 3 pulleys, and the weight of each be 1 lb., find the force capable of supporting a weight of 9 lbs.

10. If there be 3 pulleys, the weight of each being W, but no weight attached to the lowest, show that there will be equilibrium when $P : W :: 7 : 8$.

11. If there are 3 pulleys of equal weight, what must be the weight of each in order that a weight of 56 lbs. attached to the lowest may be supported by a power equal to 7 lbs. 14 oz.?

12. What must be the weight of each pulley that P may equal W, the pulleys being all of equal weight?

13. If all the pulleys except the lowest be considered weightless, and the weight of the lowest and the power be each p lbs., and the weight attached be w lbs., show that w is some odd multiple of p.

14. A man supports a weight equal to half his own weight by a system of pulleys in which the same string passes round all the pulleys, the upper block being attached to the ceiling: if there be 7 strings at the lower block, find his pressure on the floor on which he stands.

15. What weight will be supported if there be 3 pulleys in the lower block, the string being fastened to the upper block, and the weight of the lower block being equal to 3 times the power?

16. If the weight of the lower block and the power be each p lbs., and the weight attached to the lower block be w lbs., show that w is some odd or even multiple of p according as the end of the string is fastened to the upper or lower block.

17. Suppose that a power of 3 lbs. will just support a weight of 10 lbs. suspended from the lower block, the number of strings being 4, what is the weight of the lower block?

18. A power P and a weight W are in equilibrium on a system of pulleys in which all the strings are parallel and attached to a uniform bar from which the weight is suspended, the weights of the pulleys being neglected. If the number of pulleys is three, and the strings are equidistant, from what point of the bar ought the weight to be suspended that the bar may rest in a horizontal position?

19. In a system of 6 pulleys of equal weight where each pulley is attached to a string which is attached also to the weight, find the ratio which the weight of each pulley must bear to the weight supported in order that there may be equilibrium without any power being applied.

20. What is meant by saying "what is gained in power is lost in time"?

INCLINED PLANE

1. If, when P acts along the plane, $R : P :: 3 : 4$, express R and P in terms of W.

2. If W be 3 tons, find P, acting parallel to the plane, when the height of the plane is to its base as $5 : 12$.

3. Find the pressure on the plane when the height of the plane is to its base as $3 : 4$, and the weight supported is 10 lbs., the power being parallel to the plane.

4. Find the horizontal force necessary to support a body whose weight is 12 lbs. upon a plane whose base is to its length as $4 : 5$.

5. If the pressure on the plane be 2 lbs. and the power acting horizontally 1 lb., what is the weight? and what the inclination of the plane?

6. A force of 15 lbs. acting horizontally supports a weight of $5\sqrt{3}$ lbs. on an inclined plane; find the inclination of the plane to the horizon.

7. The weight supported upon an inclined plane is $2\sqrt{2}$ lbs., and the plane is inclined at half a right angle to the horizon; find the power which acting along the plane will support the weight.

8. A weight of 56 lbs. rests upon a smooth plane inclined at $45°$ to the horizon. What is the smallest horizontal force required to move it up the plane?

9. What force, acting horizontally, will sustain a weight of 12 lbs. on a plane inclined to the horizon at an angle equal to that of an equilateral triangle?

10. What force, acting horizontally, will sustain a weight of 10 lbs. on a plane inclined to the horizon at an angle equal to half of one of the angles of an equilateral triangle?

11. If the force which will support a weight when acting parallel to the plane be half that which will do so acting horizontally, find the inclination of the plane.

12. Show that if P instead of acting parallel to the plane were to make the same angle with the vertical as the pressure of the plane on the body, the pressure on the plane would be equal to P.

13. Which will support the greater weight, a power acting horizontally or the same power acting parallel to the plane?

14. A weight of 20 lbs. is supported by a string fastened to a point in an inclined plane, and the string is only just strong enough to support a weight of 10 lbs.; the inclination of the plane to the horizon being gradually increased, find when the string will break.

15. A railway train weighing 36 tons is drawn by a rope up an incline of 1 in 40; if friction produces a resistance to motion equal to 1 ton, what is the least strength of rope necessary?

16. Two unequal weights W and W' connected by a string are placed upon two smooth inclined planes, the string passing over the intersection of the planes. Find the ratio between the weights, when there is equilibrium.

17. Give a geometrical construction for determining the direction in which the power must act when it is equal to the weight, and show that if R_1 be the pressure on the plane in this case, and R the pressure when the power acts parallel to the plane, $R_1 = 2R$.

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